Brownian Fluctuations of a non-confining potential

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1. Introduction

Brownian fluctuations arise for any quantity that depends on the stochastic variables of a Brownian particle. In this study, we explore the Brownian fluctuations of a bidimensional quadratic potential that exhibits two regimes: a confining regime and a non-confining regime. We divide the total potential into two contributions and analyze the central moments and their distributions for each contribution as well as for the total potential.

The system of interest is given by a set of two Langevin equations

$$\gamma \dot{x}(t) = -kx(t) + uy(t) + \eta_x(t), \quad (1)$$
$$\gamma \dot{y}(t) = -ky(t) + ux(t) + \eta_y(t), \quad (2)$$

corresponding to the variation of the position of the particle in a $(x(t), y(t))$ position with a white noise $\eta_i(t)$.

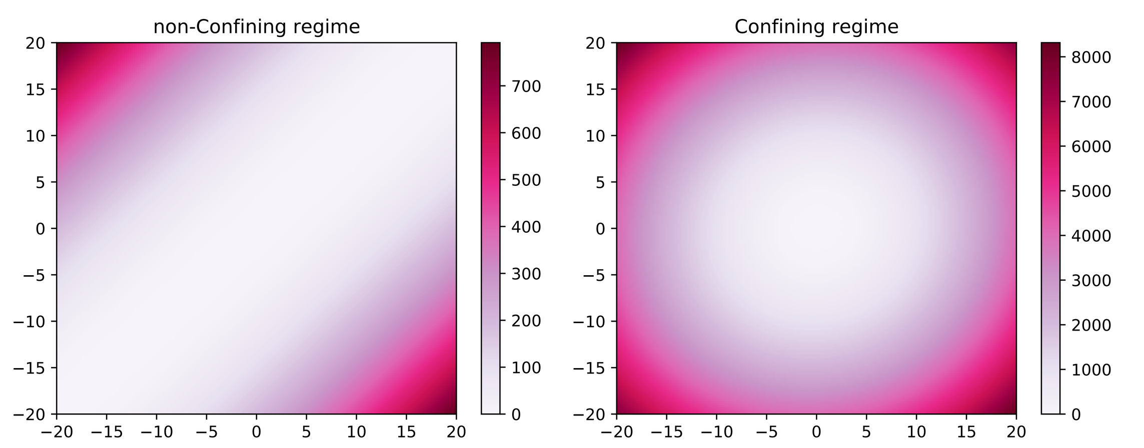

Figure 1: Two distinct regimes of the full potential, $V(x,y)$, in the left the non-confining, and in the right the confining. Note that, despite the coupling, the confining is qualitatively almost of the same shape of the harmonic potential.

2. Methodology

We use path integral formalism to calculate the conditional probability, meaning

$$P(x_\tau, y_\tau | x_0, y_0) = \int \mathcal{D}x \int \mathcal{D}y \exp\left(-\frac{1}{4\gamma T} S[x,y]\right). \quad (3)$$

The joint probability is given by

$$P(x_\tau, y_\tau, x_0, y_0) = P(x_\tau, y_\tau | x_0, y_0) P(x_0, y_0), \quad (4)$$

where $P(x_0, y_0) = \delta(x_0)\delta(y_0)$.

The conditional probability is valid for any regime of k and u, including $k < u$, which is not of interest for this work. It is noteworthy to mention that for asymptotic times

$$\lim_{\tau\to\infty} P(x_\tau, y_\tau | x_0, y_0) \to \frac{\sqrt{k^2 - u^2}}{2\pi T} \times \exp\left(-\frac{k\left(x_\tau^2 + y_\tau^2\right) - 2u x_\tau y_\tau}{2T}\right). \quad (5)$$

With the joint probability, we calculate the generating function of the distributions, that has a general form

$$Z_{\mathcal{O}}(\lambda) = \frac{\sqrt{\alpha_3}}{\sqrt{\alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3}}, \quad (6)$$

where $\alpha_i (i = 1, 2, 3)$ variates for each potential segment $\mathcal{O} = \{\Delta U, \Delta V_{nc}, \Delta V\}$. From the generating function, it is possible to calculate the central moments, and the probability distribution $P(\mathcal{O})$ is given by the Fourier transform of $Z_{\mathcal{O}}(\lambda)$.

3. Results

We calculate:

- Central moments: mean (μ), variance (σ^2), skewness (μ_3), and the curtosis excess (κ). For each distribution, we calculated the variation of the moments with the time and the factor u, with relation to k,
- Probability distributions $P(\Delta V)$ for each potential term, and for the total potential.

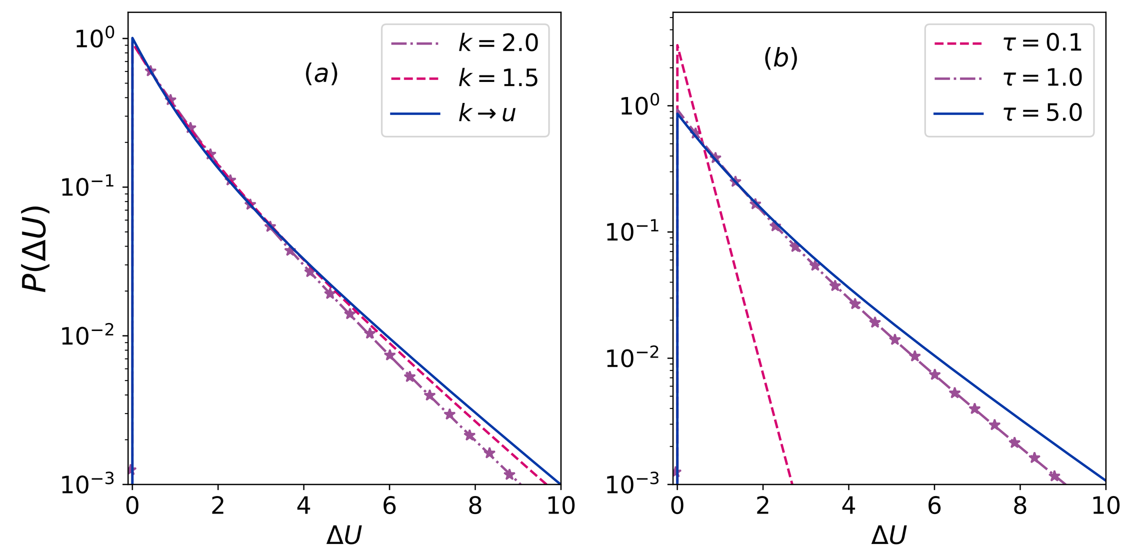

Figure 2: Probability distribution of the non-confining potential $P(\Delta U)$, varying with ΔU, and the purple stars representing our results for the simulations of Langevin equation. In (a), distributions for different values of k are depicted. In (b), the variation of $P(\Delta U)$ with ΔU for different values of time τ.

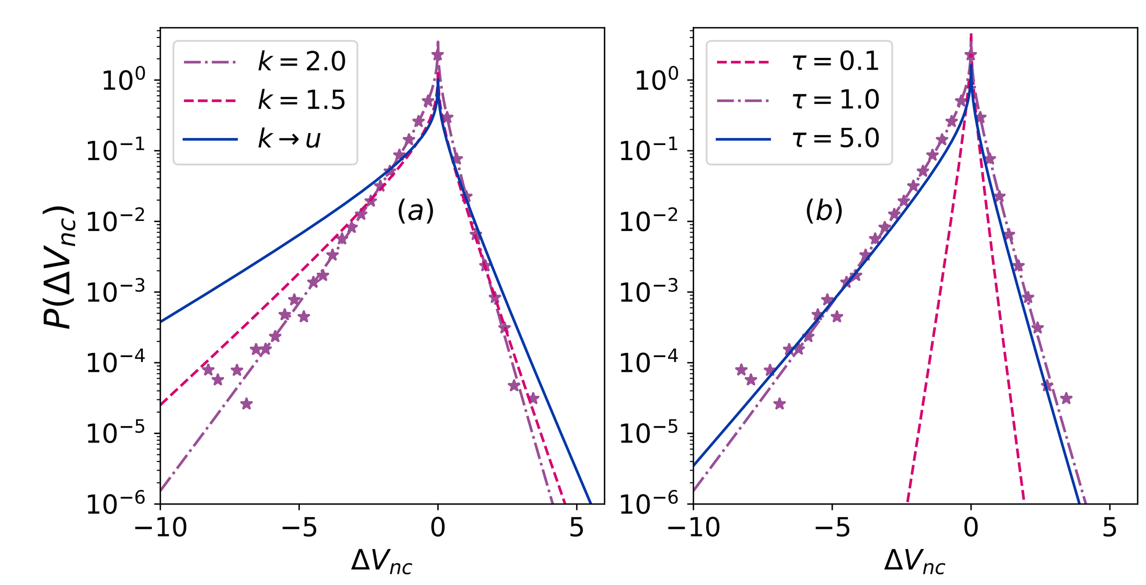

Figure 3: Probability distribution of the non-confining potential $P(\Delta V_{nc})$, varying with ΔV_{nc}, and the purple stars representing our results for the simulations of Langevin equation. In (a), distributions for different values of k are depicted. In (b), the variation of $P(\Delta V_{nc})$ with ΔV_{nc} happens for different values of time τ.

For the non-harmonic distribution, we obtained an analytical expression, that gives origin to a irreversibility ratio

$$\lim_{\tau\to\infty} \lim_{k\to u} \log\left(\frac{P(\Delta V_{nc})}{P(-\Delta V_{nc})}\right) = -\frac{2\Delta V_{nc}}{T}, \quad (7)$$

which is a mathematical consequence of the expression for $P(\Delta V_{nc})$.

We calculated the central moments for each distribution. For instance, considering the total potential ΔV:

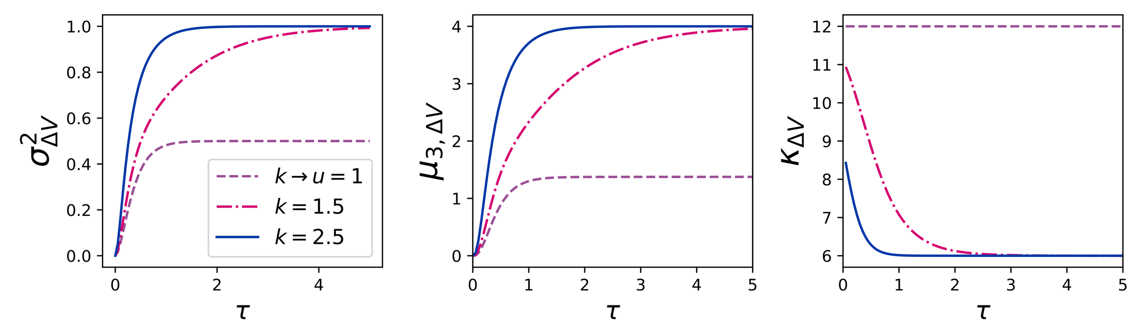

Figure 4: Variation of Central Moments of the total potential distribution with time τ. a) Evolution of $\sigma^2_{\Delta V}$ with the time τ. b) Skewness of the total distribution $\mu_{3,\Delta V}$ variating with the time τ. c) Kurtosis of the total distribution $\kappa_{\Delta V}$ variating with the time τ.

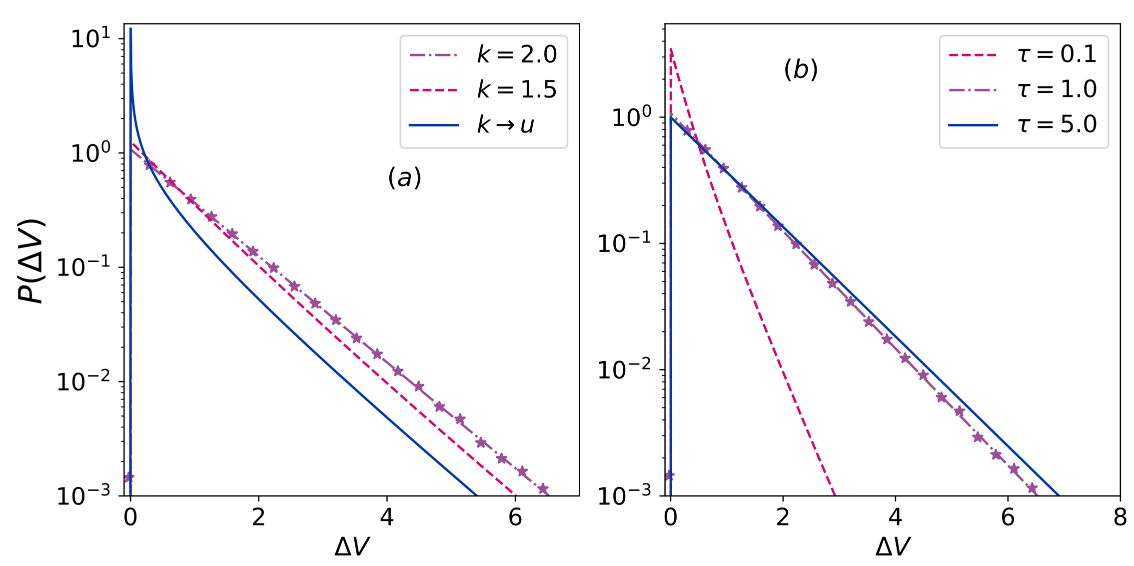

Figure 5: Variation of $P(\Delta V)$ with ΔV, with purple stars representing our results for the simulations of Langevin equation. In (a), $P(\Delta V)$ is depicted variating with ΔV, for different values of k. In (b), $P(\Delta V)$ is depicted variating with ΔV, for different times τ.

4. Conclusions

We calculated the Brownian fluctuations of a quadratic potential, which exhibits two regimes, one of confinement and one of non-confinement. These fluctuations are inherent to the system, that consists of a Brownian particle in two dimensions, with motion starting at the origin. Remarkably, as a mathematical consequence of $P(\Delta V_{nc})$, in the asymptotic limit of the non-confining limit $k \to u$ the log ratio between $\frac{P(\Delta V_{nc})}{P(-\Delta V_{nc})}$ gives origin to a irreversibility ratio that is identical to the Crooks' fluctuation theorem. This is purely a mathematical consequence to the form of the potential.

References

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