a Lorentz invariant velocity distribution for a relativistic gas

EMFC, Carlos Cedeno, Ivano D. Soares, Constantino Tsallis
CBPf - INCT-SC

statistical mechanics for complexity
a celebration of the 80th birthday of Constantino Tsallis
2023

first students

- four students - from UnB to CBPF:
  - Ivan Frederico Lupiano Dias (UEL/PR),
  - Sydney Francisco Machado (UFES/ES)
  - Darly Machado Henriques da Silva (CNPq e MCT/DF) e
  - Evaldo M. F. Curado (CBPF).

  at CBPF,

  - Aglaé Cristina Navarro de Magalhães (CBPF)

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Ivan

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PHYSICAL REVIEW B VOLUME 21, NUMBER 7 1 APRIL 1980

Approximate critical surface of the bond-mixed square-lattice Ising model

Silvio V. F. Levy, Constantino Tsallis, and Evaldo M. F. Curado
Centro Brasileiro de Pesquisas Físicas, Conselho Nacional de Desenvolvimento Científico e Tecnológico, Avenida Wenceslau Braz 71, Rio de Janeiro, Brazil
(Received 19 September 1979)

The critical surface of the quenched bond-mixed square-lattice spin-$\frac{1}{2}$ first-neighbor-interaction ferromagnetic Ising model (with exchange interactions J_1 and J_2) has been investigated. Through renormalization group and heuristical procedures, a very accurate [error inferior to 3×10^{-4} in the variables $t_i \equiv \tanh(J_i/k_BT)$] approximate *numerical* proposal for all points of this surface is presented. This proposal simultaneously satisfies *all* the available exact results concerning the surface, namely $p_c = \frac{1}{2}$, $t_c = \sqrt{2} - 1$, both limiting slopes in these points, and $t_2 = (1 - t_1)/(1 + t_1)$ for $p = \frac{1}{2}$. Furthermore an *analytic* approximation [namely, $(1-p)\ln(1+t_1) + p\ln(1+t_2) = \frac{1}{2}\ln 2$] is also proposed. In what concerns the available exact results, it only fails in reproducing one of the two limiting slopes, where there is an error of 1% in the derivative: These facts result in an estimated error less than 10^{-3} (in the t variables) for any point in the surface.

PHYSICAL REVIEW B VOLUME 23, NUMBER 3 1 FEBRUARY 1981

Extrapolated renormalization-group calculation of the surface tension in square-lattice Ising model

Evaldo M. F. Curado, Constantino Tsallis, and Silvio V. F. Levy
Centro Brasileiro de Pesquisas Físicas, Av. Wenceslau Braz 71, Rio de Janeiro, Brazil

Mario J. de Oliveira
Instituto de Física, Universidade de São Paulo, Cidade Universitária, São Paulo, Brazil
(Received 20 June 1980)

By using self-dual clusters (whose sizes are characterized by the numbers $b = 2, 3, 4, 5$), we calculate within a real-space renormalization-group framework, the *longitudinal* surface tension of the square-lattice first-neighbor $\frac{1}{2}$-spin ferromagnetic Ising model. The exact critical temperature T_c is recovered for any value of b; the exact asymptotic behavior of the surface tension in the limit of low temperatures is *analytically* recovered; the approximate correlation-length critical exponents monotonically tend towards the exact value $\nu = 1$ (which, for two dimensions, coincides with the surface-tension critical exponent μ) for increasingly large cells; the same behavior is noticed in the approximate values for the surface-tension amplitude in the limit $T \to T_c$. We develop four different numerical procedures for extrapolating to $b \to \infty$ the renormalization-group (RG) results for the surface tension, and quite satisfactory agreement is obtained with Onsager's exact expression (error varying from zero to a few percent on the *whole* temperature domain). Furthermore, we compare the set of RG surface tensions with a set of *biased* surface tensions (associated with appropriate misfit seams) and find only fortuitous coincidence among them.

A Lorentz-invariant velocity distribution for a relativistic gas

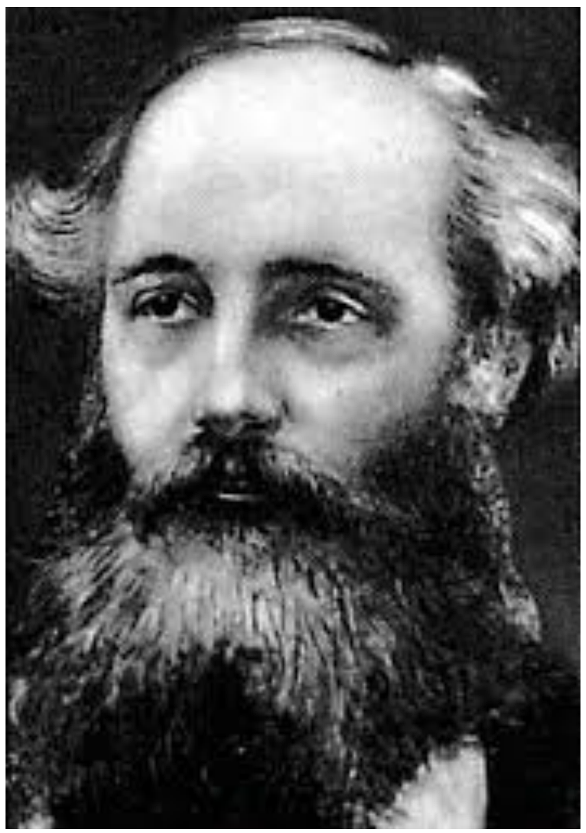
Evaldo M F Curado, Ivano Damião Soares, Carlos E. C. Montaña and Constantino Tsallis (CBPF, Rio de Janeiro, Brazil)

Tsallis 80 - INCT-SC 2023

outline

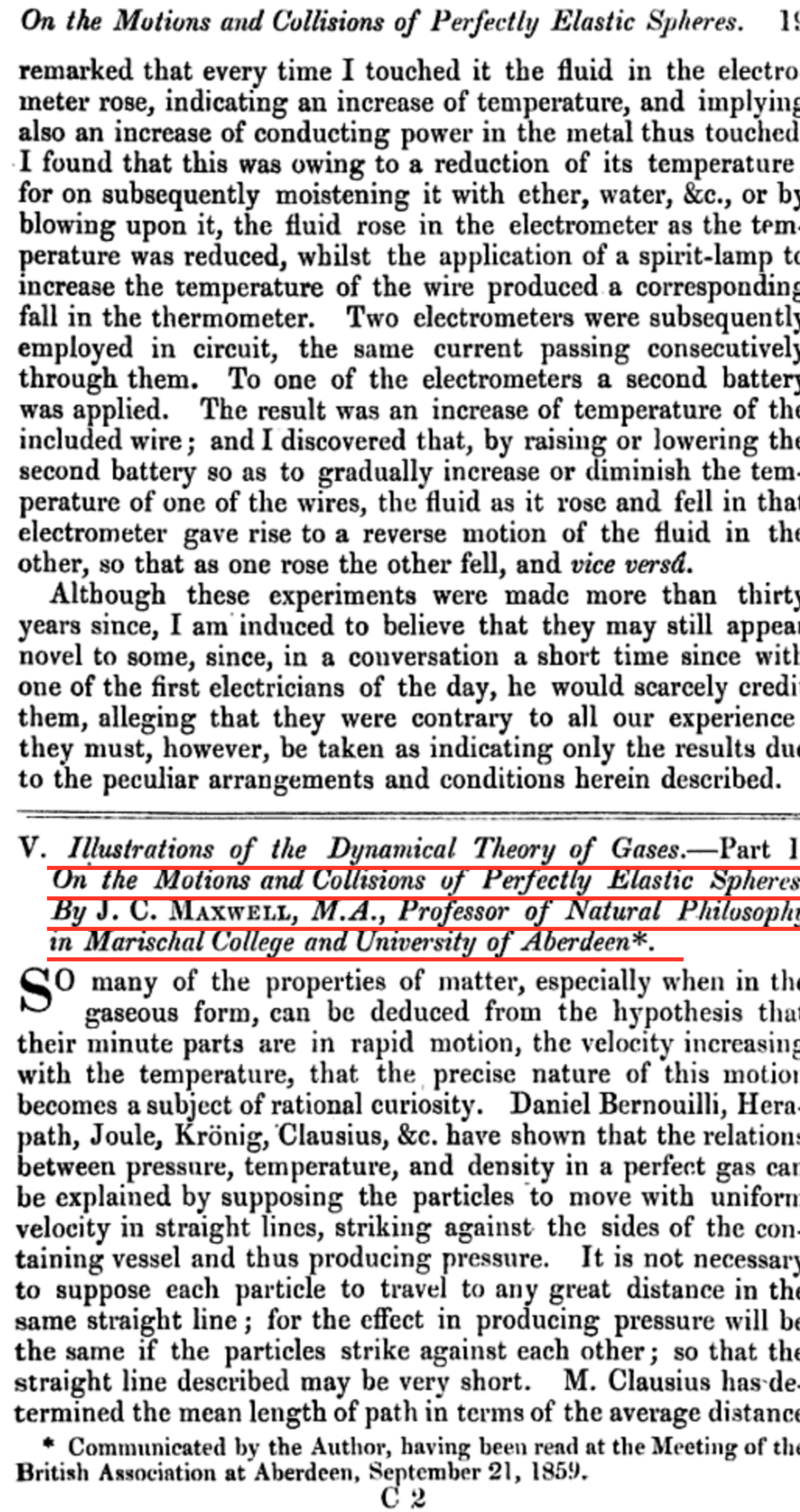
- Maxwell and Jüttner distribution
- Jüttner is not LI
- LI velocity distribution
- comparison - molecular dynamical simulations

Maxwell (1859-1860)



- first time probability enters in a physical theory

Philosophical Magazine XIX (1860) 19-32



remarked that every time I touched it the fluid in the electrometer rose, indicating an increase of temperature, and implying also an increase of conducting power in the metal thus touched. I found that this was owing to a reduction of its temperature; for on subsequently moistening it with ether, water, &c., or by blowing upon it, the fluid rose in the electrometer as the temperature was reduced, whilst the application of a spirit-lamp to increase the temperature of the wire produced a corresponding fall in the thermometer. Two electrometers were subsequently employed in circuit, the same current passing consecutively through them. To one of the electrometers a second battery was applied. The result was an increase of temperature of the included wire; and I discovered that, by raising or lowering the second battery so as to gradually increase or diminish the temperature of one of the wires, the fluid as it rose and fell in that electrometer gave rise to a reverse motion of the fluid in the other, so that as one rose the other fell, and *vice versâ*.

Although these experiments were made more than thirty years since, I am induced to believe that they may still appear novel to some, since, in a conversation a short time since with one of the first electricians of the day, he would scarcely credit them, alleging that they were contrary to all our experience; they must, however, be taken as indicating only the results due to the peculiar arrangements and conditions herein described.

V. *Illustrations of the Dynamical Theory of Gases.*—Part I. *On the Motions and Collisions of Perfectly Elastic Spheres. By* J. C. Maxwell, *M.A., Professor of Natural Philosophy in Marischal College and University of Aberdeen*.*

So many of the properties of matter, especially when in the gaseous form, can be deduced from the hypothesis that their minute parts are in rapid motion, the velocity increasing with the temperature, that the precise nature of this motion becomes a subject of rational curiosity. Daniel Bernouilli, Herapath, Joule, Krönig, Clausius, &c. have shown that the relations between pressure, temperature, and density in a perfect gas can be explained by supposing the particles to move with uniform velocity in straight lines, striking against the sides of the containing vessel and thus producing pressure. It is not necessary to suppose each particle to travel to any great distance in the same straight line; for the effect in producing pressure will be the same if the particles strike against each other; so that the straight line described may be very short. M. Clausius has determined the mean length of path in terms of the average distance

* Communicated by the Author, having been read at the Meeting of the British Association at Aberdeen, September 21, 1859.

plane AGN, making N G a = N G A, and G a = G A and G b = G B; then by Prop. I. G a and G b will be the velocities relative to G; and compounding these with O G, we have O a and O b for the true velocities after impact.

By Prop. II. all directions of the line a G b are equally probable. It appears therefore that the velocity after impact is compounded of the velocity of the centre of gravity, and of a velocity equal to the velocity of the sphere relative to the centre of gravity, which may with equal probability be in any direction whatever.

If a great many equal spherical particles were in motion in a perfectly elastic vessel, collisions would take place among the particles, and their velocities would be altered at every collision; so that after a certain time the *vis viva* will be divided among the particles according to some regular law, the average number of particles whose velocity lies between certain limits being ascertainable, though the velocity of each particle changes at every collision.

Prop. IV. To find the average number of particles whose velocities lie between given limits, after a great number of collisions among a great number of equal particles.

Let N be the whole number of particles. Let x, y, z be the components of the velocity of each particle in three rectangular directions, and let the number of particles for which x lies between x and $x+dx$ be $Nf(x)dx$, where $f(x)$ is a function of x to be determined.

The number of particles for which y lies between y and $y+dy$ will be $Nf(y)dy$; and the number for which z lies between z and $z+dz$ will be $Nf(z)dz$, where f always stands for the same function.

Now the existence of the velocity x does not in any way affect that of the velocities y or z, since these are all at right angles to each other and independent, so that the number of particles whose velocity lies between x and $x+dx$, and also between y and $y+dy$, and also between z and $z+dz$, is

$$Nf(x)\,f(y)\,f(z)\,dx\,dy\,dz.$$

If we suppose the N particles to start from the origin at the same instant, then this will be the number in the element of volume $(dx\,dy\,dz)$ after unit of time, and the number referred to unit of volume will be

$$Nf(x)\,f(y)\,f(z).$$

But the directions of the coordinates are perfectly arbitrary, and therefore this number must depend on the distance from the origin alone, that is

$$f(x)\,f(y)\,f(z)=\phi(x^2+y^2+z^2).$$

Solving this functional equation, we find

$$f(x)=\mathrm{C}e^{\mathrm{A}x^2}, \quad \phi(r^2)=\mathrm{C}^3e^{\mathrm{A}r^2}.$$

If we make A positive, the number of particles will increase with the velocity, and we should find the whole number of particles infinite. We therefore make A negative and equal to $-\frac{1}{\alpha^2}$, so that the number between x and $x+dx$ is

$$\mathrm{NC}e^{-\frac{x^2}{\alpha^2}}dx.$$

Integrating from $x=-\infty$ to $x=+\infty$, we find the whole number of particles,

$$\mathrm{NC}\sqrt{\pi}\alpha=\mathrm{N}, \quad \therefore \mathrm{C}=\frac{1}{\alpha\sqrt{\pi}},$$

$f(x)$ is therefore

$$\frac{1}{\alpha\sqrt{\pi}}e^{-\frac{x^2}{\alpha^2}}.$$

Whence we may draw the following conclusions:—

1st. The number of particles whose velocity, resolved in a certain direction, lies between x and $x+dx$ is

$$\mathrm{N}\frac{1}{\alpha\sqrt{\pi}}e^{-\frac{x^2}{\alpha^2}}dx. \quad \ldots \ldots \quad (1)$$

2nd. The number whose actual velocity lies between v and $v+dv$ is

$$\mathrm{N}\frac{4}{\alpha^3\sqrt{\pi}}v^2e^{-\frac{v^2}{\alpha^2}}dv. \quad \ldots \ldots \quad (2)$$

3rd. To find the mean value of v, add the velocities of all the particles together and divide by the number of particles; the result is

$$\text{mean velocity} = \frac{2\alpha}{\sqrt{\pi}}. \quad \ldots \ldots \quad (3)$$

4th. To find the mean value of v^2, add all the values together and divide by N,

$$\text{mean value of } v^2=\tfrac{3}{2}\alpha^2. \quad \ldots \ldots \quad (4)$$

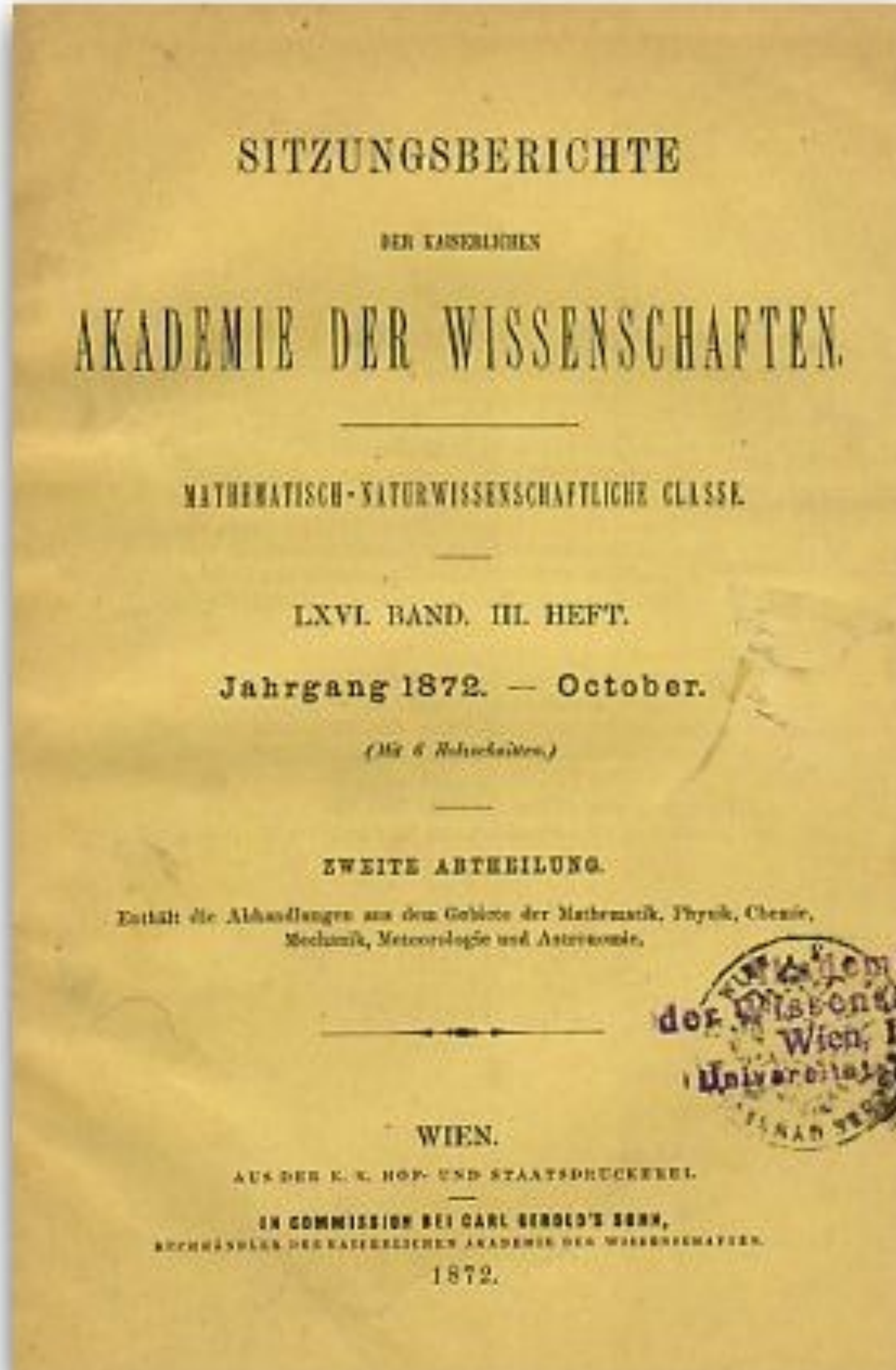
This is greater than the square of the mean velocity, as it ought to be.

It appears from this proposition that the velocities are distributed among the particles according to the same law as the errors are distributed among the observations in the theory of the "method of least squares." The velocities range from 0 to ∞, but the number of those having great velocities is comparatively small. In addition to these velocities, which are in all directions equally, there may be a general motion of translation

$$f(\vec{v}^{\,2}) \propto \exp\left(-\frac{m\vec{v}^{\,2}}{2kT}\right)$$

H-theorem

Boltzmann 1872



SITZUNGSBERICHTE

DER KAISERLICHEN

AKADEMIE DER WISSENSCHAFTEN.

MATHEMATISCH-NATURWISSENSCHAFTLICHE CLASSE.

LXVI. BAND. III. HEFT.

Jahrgang 1872. — October.

(Mit 6 Holzschnitten.)

ZWEITE ABTHEILUNG.

Enthält die Abhandlungen aus dem Gebiete der Mathematik, Physik, Chemie, Mechanik, Meteorologie und Astronomie.

WIEN.

AUS DER K. K. HOF- UND STAATSDRUCKEREI.

IN COMMISSION BEI CARL GEROLD'S SOHN,

BUCHHÄNDLER DER KAISERLICHEN AKADEMIE DER WISSENSCHAFTEN.

1872.

Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen.

Von **Ludwig Boltzmann** in Graz.

(Mit 4 Holzschnitten.)

(Vorgelegt in der Sitzung am 10. October 1872.)

Die mechanische Wärmetheorie setzt voraus, dass sich die Moleküle der Gase keineswegs in Ruhe, sondern in der lebhaftesten Bewegung befinden. Wenn daher auch der Körper seinen Zustand gar nicht verändert, so wird doch jedes einzelne seiner Moleküle seinen Bewegungszustand beständig verändern, und

$$
\begin{aligned}
\frac{\partial f(x,t)}{\partial t} &= \int_0^\infty \int_0^{x+x'} \left[\frac{f(\xi,t)}{\sqrt{\xi}} \frac{f(x+x'-\xi,t)}{\sqrt{x+x'-\xi}} \right. \\
&- \left. f(x,t)\sqrt{x}\frac{f(x',t)}{\sqrt{x'}} \right] \\
&\times \sqrt{xx'}\psi(x,x',\xi)dx'd\xi.
\end{aligned}
$$

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$$

$f_M(x,t) = C\sqrt{x}e^{-hx}$ ⟶ $\partial f_M/\partial t = 0$

$$\frac{\partial f(x,t)}{\partial t} = \int_0^\infty \int_0^{x+x'} \left[\frac{f(\xi,t)}{\sqrt{\xi}} \frac{f(x+x'-\xi,t)}{\sqrt{x+x'-\xi}} - f(x,t)\sqrt{x}\frac{f(x',t)}{\sqrt{x'}} \right] \times \sqrt{xx'}\psi(x,x',\xi)dx'd\xi.$$

$$f_M(x,t) = C\sqrt{x}e^{-hx} \longrightarrow \partial f_M/\partial t = 0$$

gegenwärtigen Untersuchung bildet, des Satzes nämlich, dass die Grösse

$$E = \int_0^\infty f(x,t)\left\{\log\left[\frac{f(x,t)}{\sqrt{x}}\right] - 1\right\}dx \qquad 17) \quad \frac{dE}{dt} \leq 0$$

niemals zunehmen kann, wenn die in dem bestimmten Integrale

p. 295 - generalize the entropy concept to non-equilibrium states

$$\frac{\partial f(x,t)}{\partial t} = \int_0^\infty \int_0^{x+x'} \left[\frac{f(\xi,t)}{\sqrt{\xi}} \frac{f(x+x'-\xi,t)}{\sqrt{x+x'-\xi}} - f(x,t)\sqrt{x}\frac{f(x',t)}{\sqrt{x'}} \right] \times \sqrt{xx'}\psi(x,x',\xi)dx'd\xi.$$

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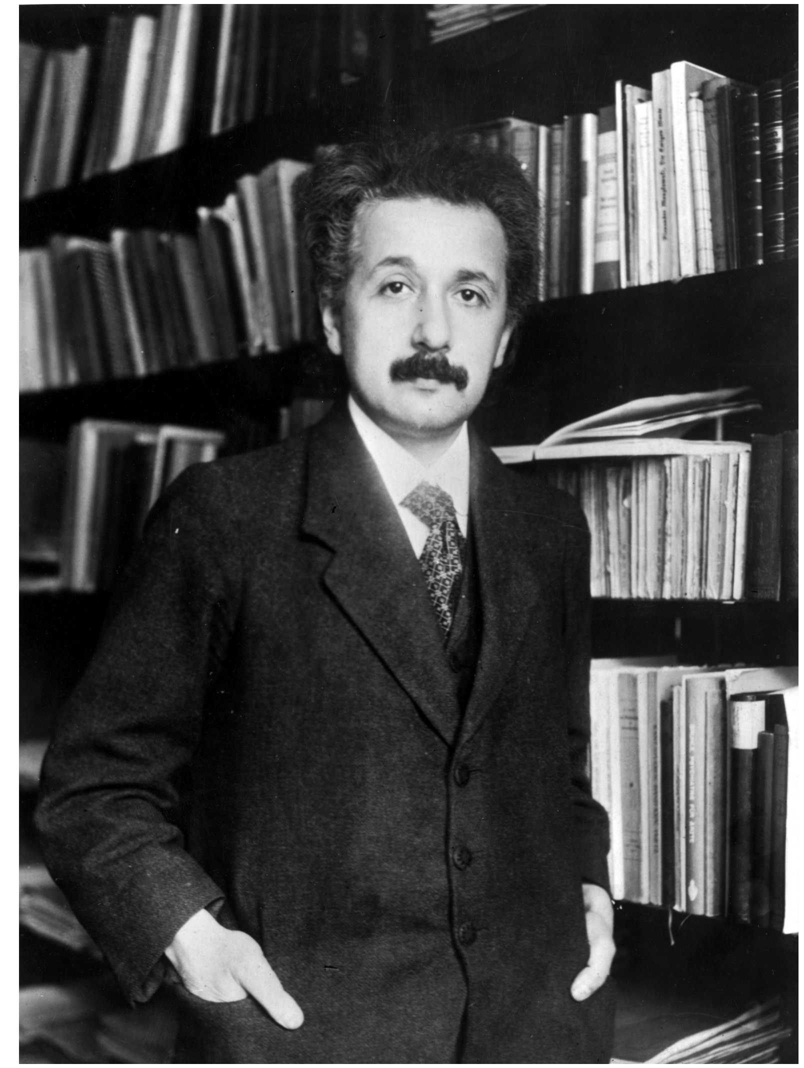
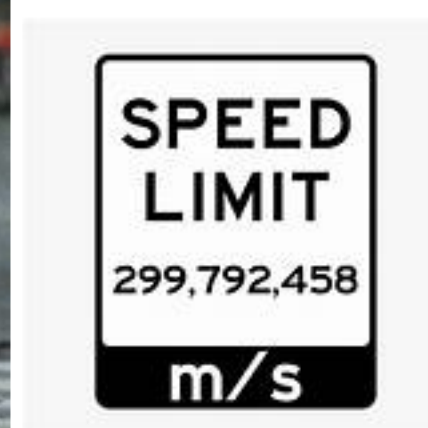
p. 295 - generalize the entropy concept to non-equilibrium states

special relativity (1905):
maximum speed = c





Jüttner distribution

Ann. Phys. (Leipzig) 18 (1911) 856

2. *Das Maxwellsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie;*

von Ferencz Jüttner.

Um die Entropie S eines idealen einatomigen Gases in einem gegebenen Zustande mittels ihrer allgemeinen Definition

$$S = k \log W, \tag{1}$$

worin W die Wahrscheinlichkeit des Zutandes ist, zu berechnen und daraus die thermodynamischen Eigenschaften des Gases abzuleiten, muß man eine bestimmte Mechanik zugrunde legen. Man wählte nun bisher stets die Newtonsche Mechanik. Vom Standpunkte des Relativprinzips von A. Einstein ist diese aber bekanntlich nur ein Grenzfall einer allgemeineren Mechanik, und es dürfte von Interesse sein, die Entropie eines solchen ruhenden Gases, sowie mit ihrer Hilfe das Gesetz der Geschwindigkeitsverteilung im Gleichgewichtszustand unter Annahme der Relativitätsmechanik zu ermitteln; daran mögen sich dann die Folgerungen über die Zustandsgleichung und die Abhängigkeit der Energie, Entropie und freien Energie von der Temperatur (spezifische Wärmen, adiabatische Gleichungen) anschließen.

Es ist mir eine angenehme Pflicht, Hrn. Geheimrat Planck für die freundliche Anregung zu dieser Arbeit und seine wohlwollenden Ratschläge meinen wärmsten Dank auszusprechen.

§ 1. Die Entropie eines ruhenden idealen einatomigen Gases in einem beliebigen Zustande.

Von den Begriffen der Relativitätsmechanik[1]) kommen für die kinetische Gastheorie erstlich in Betracht die Impulskomponenten ξ, η, ζ eines mit der Geschwindigkeit

$$q = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

1) M. Planck, Verh. d. Deutsch. Physik. Ges. 8. p. 136—141. 1906.

Jüttner distribution

$$d\omega = dx\,dy\,dz\,d\rho\,d\varpi\,d\xi \qquad \rho = \gamma(v)\dot{x}\,, \varpi = \gamma(v)\dot{y}\,, \xi = \gamma(v)\dot{z}$$

Jüttner distribution

$$d\omega = dx\,dy\,dz\,d\rho\,d\varpi\,d\xi \qquad \rho = \gamma(v)\dot{x}\,,\varpi = \gamma(v)\dot{y}\,,\xi = \gamma(v)\dot{z}$$

MaxEnt

$$S = -k\int F\ln F dx\,dy\,dz\,d\rho\,dw\,d\xi\,dv$$

$$E = m_0c^2\int \gamma(v)F dx\,dy\,dz\,d\rho\,dw\,d\xi\,dv$$

Jüttner distribution

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$$dN = F d\omega = \alpha e^{-\beta\sqrt{1+\frac{\rho^2+\varpi^2+\xi^2}{c^2}}}dx\,dy\,dz\,d\rho\,d\varpi\,d\xi$$

Jüttner distribution

$$d\omega = dx\, dy\, dz\, d\rho\, d\varpi\, d\xi \qquad \rho = \gamma(v)\dot{x}\,, \varpi = \gamma(v)\dot{y}\,, \xi = \gamma(v)\dot{z}$$

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$$dN = f d\sigma \qquad d\sigma = dx\, dy\, dz\, d\dot{x}\, d\dot{y}\, d\dot{z}$$

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$$d\omega = \frac{\partial(\rho,\varpi,\xi)}{\partial(\dot{x},\dot{y},\dot{z})}d\sigma = \gamma(v)^5 d\sigma$$ (three-dimension)

$$\gamma = 1/\sqrt{1-v^2/c^2}$$

Jüttner distribution

$$d\omega = dx\, dy\, dz\, d\rho\, d\varpi\, d\xi \qquad \rho = \gamma(v)\dot{x}\,, \varpi = \gamma(v)\dot{y}\,, \xi = \gamma(v)\dot{z}$$

MaxEnt

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$$d\omega = \frac{\partial(\rho, \varpi, \xi)}{\partial(\dot{x}, \dot{y}, \dot{z})} d\sigma = \gamma(v)^5 d\sigma$$

(three-dimension)

$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

$$f = \gamma^5(v)\, F$$

Jüttner distribution

- in one dimension

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$$Z_J = 2m_0 K_1(m_0 c^2/(kT_J))$$

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- in d dimensions

$$f_J(v, m, \beta_J) = \frac{1}{Z_J} m_0^d\, \gamma(\mathbf{v})^{2+d}\, \exp[-\beta_J m_0 c^2 \gamma(\mathbf{v})]$$

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- relativistic velocity space has a negative curvature - Lobachevsky space

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- correct factor: $\gamma(\mathbf{v})^{1+d}$

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two problems with Jüttner

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two problems with Jüttner

Jüttner is not LI

alternative distributions

alternative distributions

modified Jüttner distribution

$$f_{MJ}(v, m_0, \beta_J) = \frac{m_0^{d-1}}{Z_{MJ}} \gamma(\mathbf{v})^{1+d} \exp[-\beta_J m_0 c^2 \gamma(\mathbf{v})]$$

numerical simulations - Jüttner

PRL 99, 170601 (2007) PHYSICAL REVIEW LETTERS week ending 26 OCTOBER 2007

Thermal Equilibrium and Statistical Thermometers in Special Relativity

David Cubero,[1] Jesús Casado-Pascual,[1] Jörn Dunkel,[2] Peter Talkner,[2] and Peter Hänggi[2]

[1]*Física Teórica, Universidad de Sevilla, Apartado de Correos 1065, Sevilla 41080, Spain*

[2]*Institut für Physik, Universität Augsburg, Theoretische Physik I, Universitätsstraße, D-86135 Augsburg, Germany*

(Received 23 May 2007; published 22 October 2007)

There is an intense debate in the recent literature about the correct generalization of Maxwell's velocity distribution in special relativity. The most frequently discussed candidate distributions include the Jüttner function as well as modifications thereof. Here we report results from fully relativistic one-dimensional molecular dynamics simulations that resolve the ambiguity. The numerical evidence unequivocally favors the Jüttner distribution. Moreover, our simulations illustrate that the concept of "thermal equilibrium" extends naturally to special relativity only if a many-particle system is spatially confined. They make evident that "temperature" can be statistically defined and measured in an observer frame independent way.

numerical simulations - Jüttner

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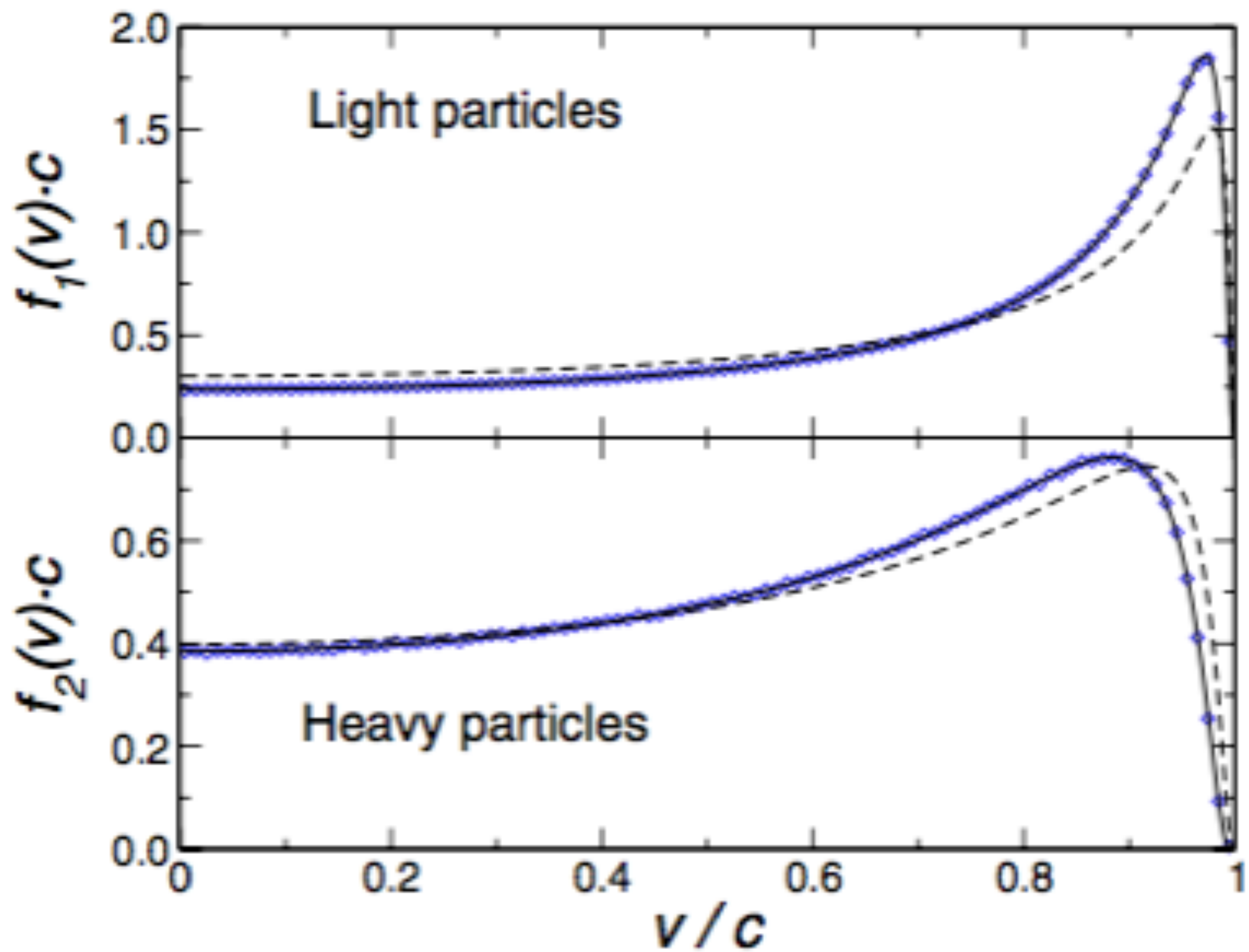
n=10 000
m2=2 m1



a puzzle: Jüttner is not LI but fits well with molecular dynamics simulations

our proposal: a LI distribution - 1D

$$v = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}$$

our proposal: a LI distribution - 1D

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \quad \longrightarrow \quad \frac{1 + v/c}{1 - v/c} = \Big(\frac{1 + v_1/c}{1 - v_1/c}\Big)\Big(\frac{1 + v_2/c}{1 - v_2/c}\Big)$$

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$$\sigma_i \equiv \frac{1}{2}\ln\left(\frac{1+v_i/c}{1-v_i/c}\right) = \tanh^{-1}(v_i/c)\ , \quad \sigma_i \in (-\infty, \infty)$$

rapidity

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$$\Rightarrow \quad \sigma = \sum_i \sigma_i$$

- σ_i - random, with zero mean

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$$s \equiv \sigma/\sqrt{N} \quad \Rightarrow \quad P(s)ds = C_1 e^{-\beta s^2} ds$$ (CLT)

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$$\boxed{P(\sigma)d\sigma = C_1 e^{-\frac{\beta}{N}\sigma^2} \frac{d\sigma}{\sqrt{N}}}$$

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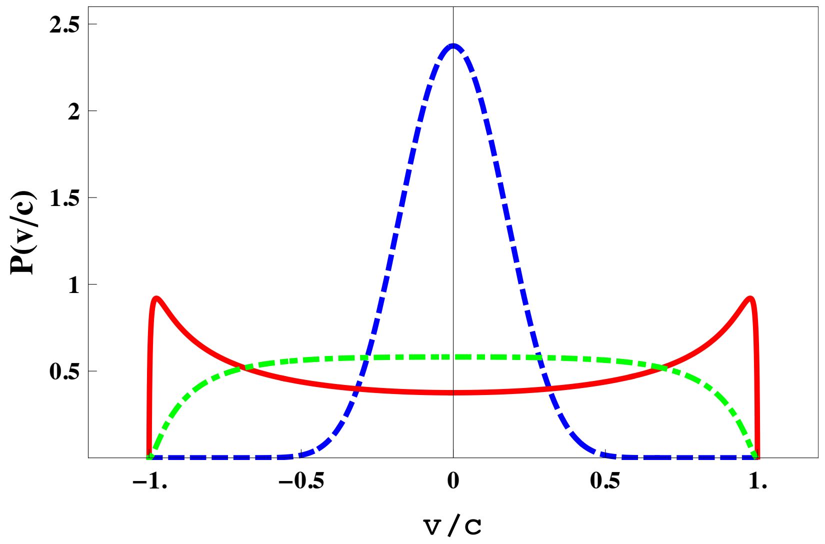

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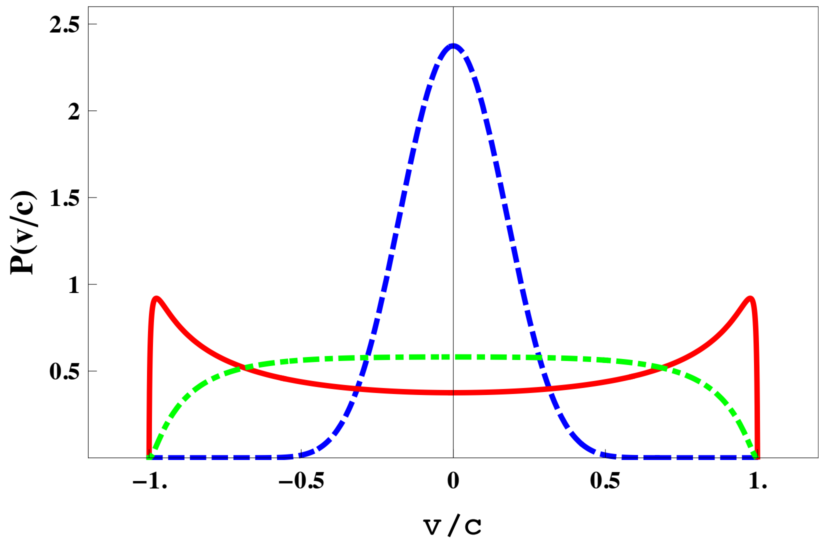

- LI velocity distribution

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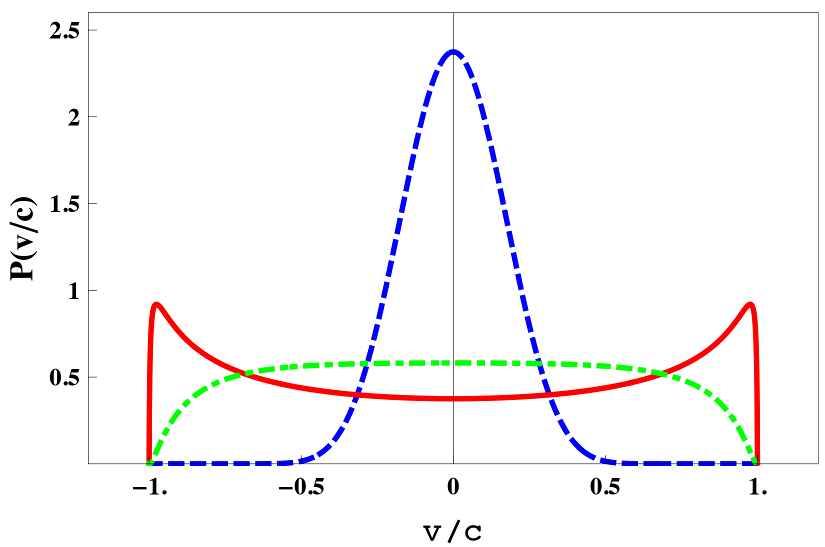

- LI velocity distribution

$$\beta = \frac{m_0 c^2}{2k_B T} \qquad v/c \to 0 \qquad \mathrm{LI} \to \mathrm{MB}$$

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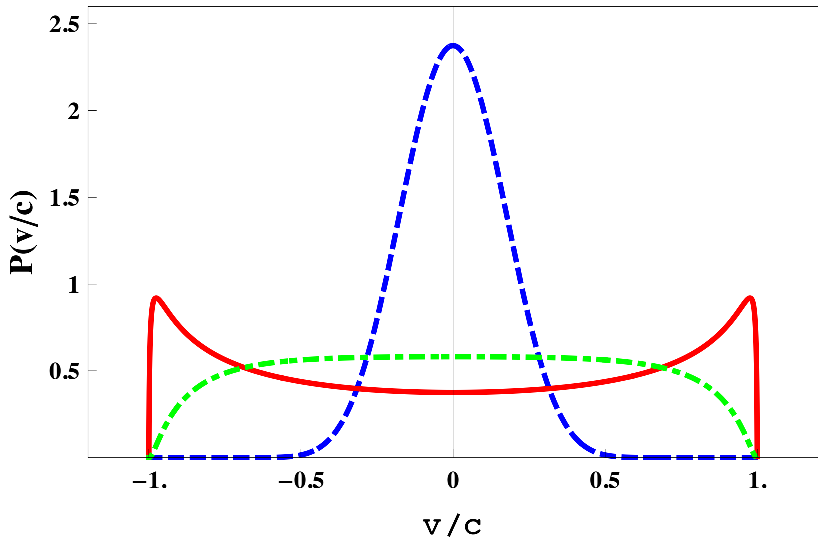

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Lorentz
invariant

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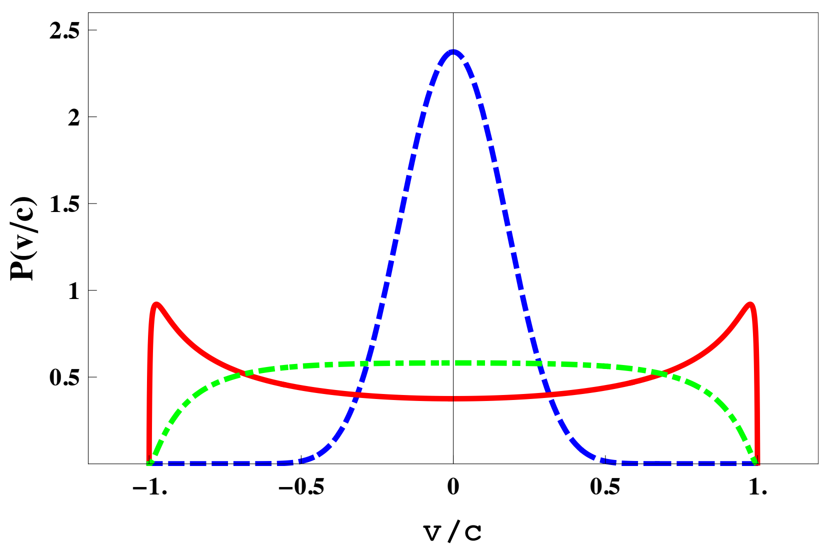

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$$P(v) = 0 \quad \text{for} \quad |v| > c$$

T
Lorentz
invariant

L. MARQUES, J. CLEYMANS, AND A. DEPPMAN

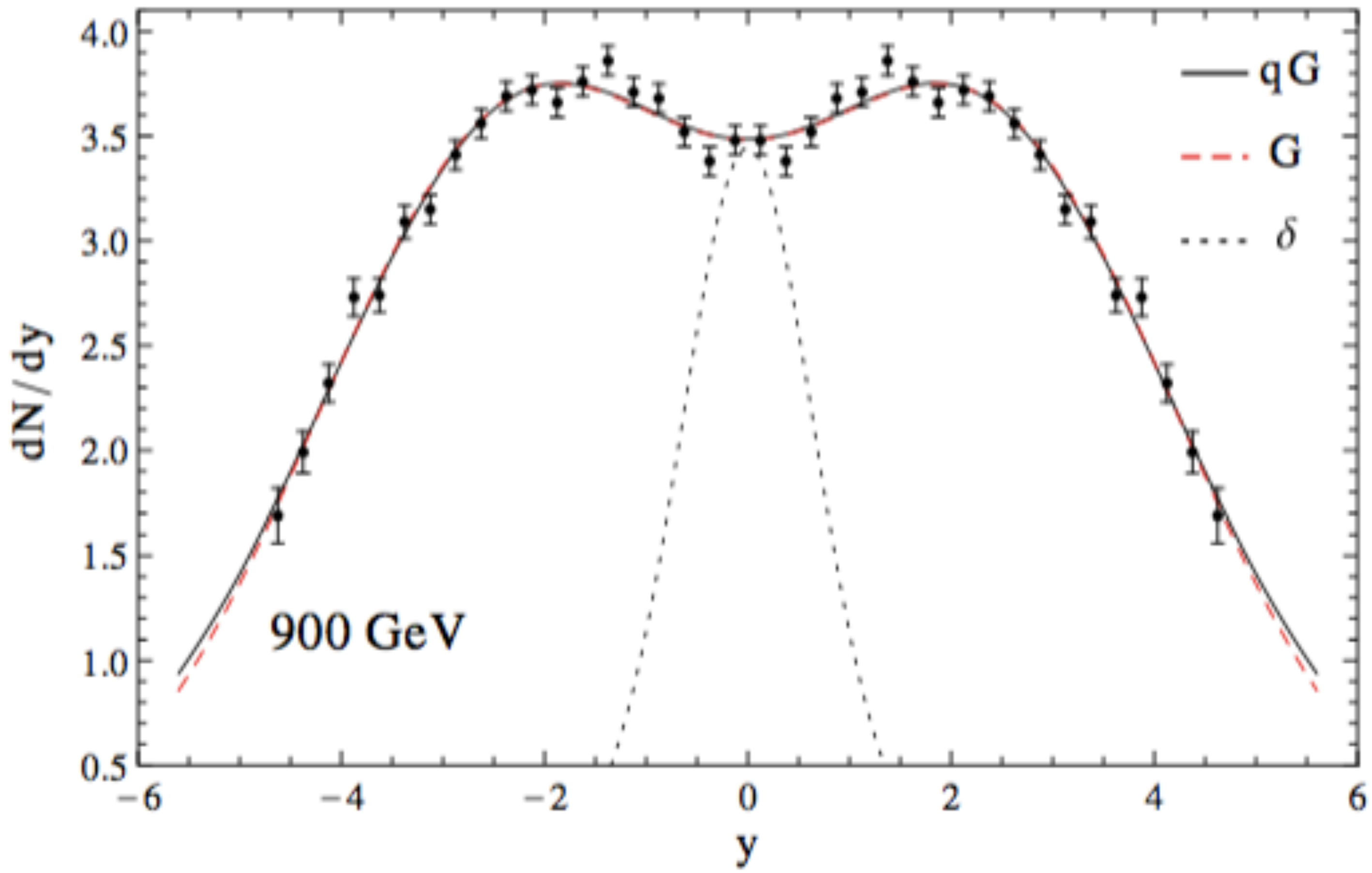
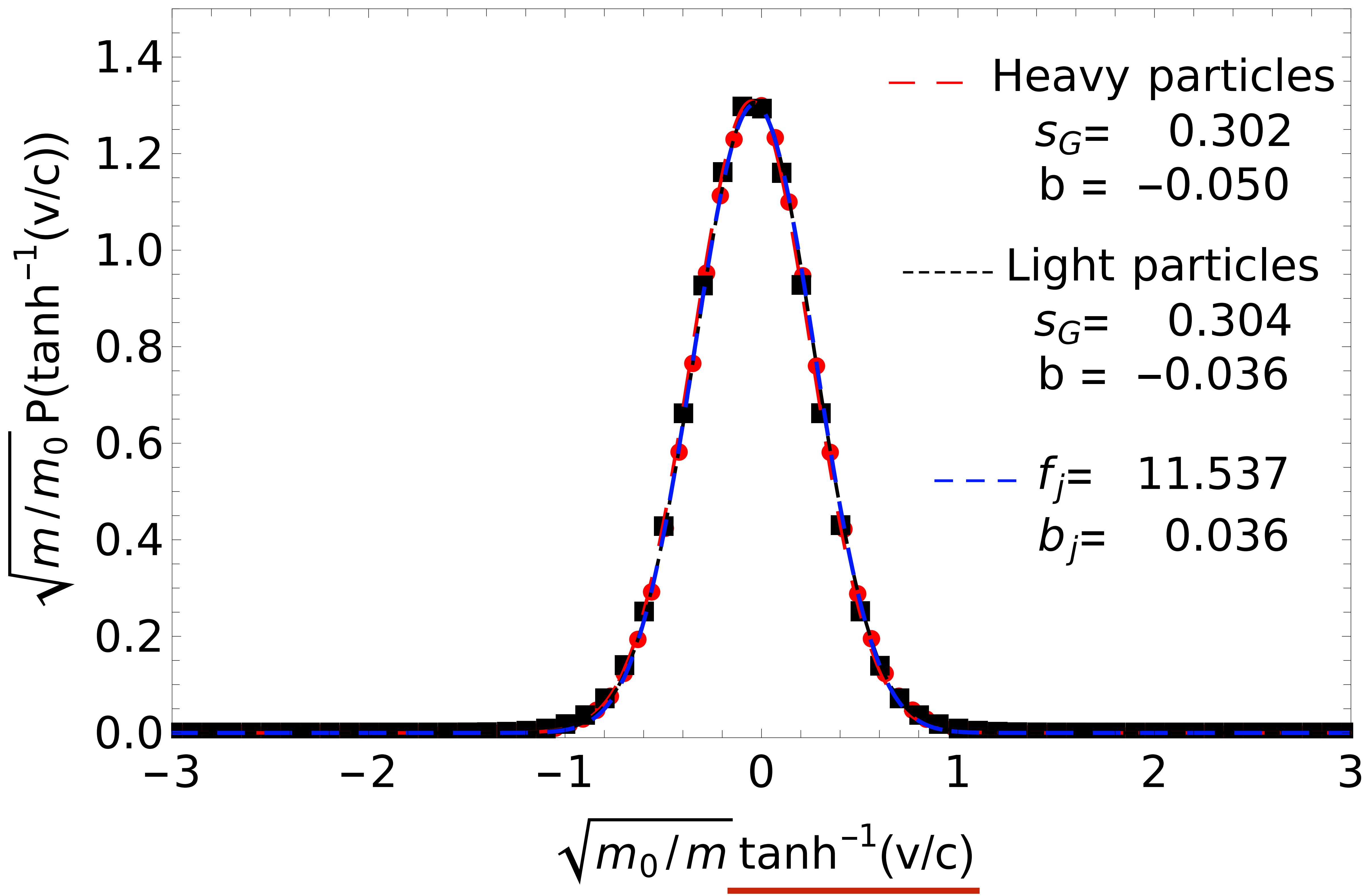
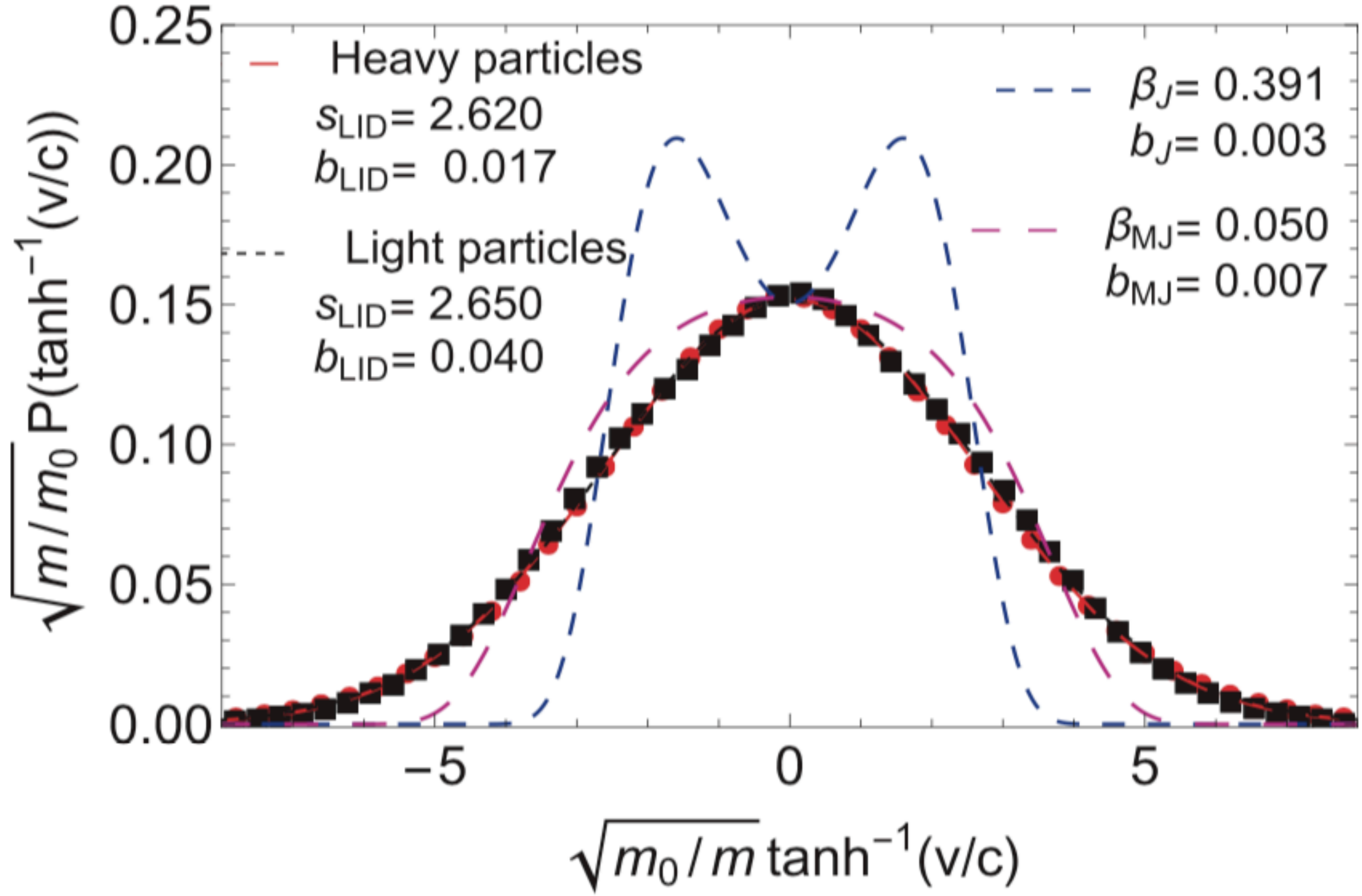


FIG. 4 (color online). Fit using Eq. (6) to experimental data for rapidity distribution from pp collision at 900 GeV from the UA5 Collaboration [31]. Data for $y < 0$ are obtained symmetrically

PRD 91 (2015) 054025



Heavy particles
s_G= 0.302
b = −0.050
Light particles
s_G= 0.304
b = −0.036
f_j= 11.537
b_j= 0.036
$\sqrt{m/m_0}\,P(\tanh^{-1}(v/c))$
$\sqrt{m_0/m}\,\tanh^{-1}(v/c)$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
−3
−2
−1
0
1
2
3



Jüttner -> $$p_J(\sigma)d\sigma = \frac{m_0}{Z_J}\ \cosh\sigma\ \exp\left(-\frac{m_0c^2}{k_BT_J}\cosh\sigma\right)\,d\sigma$$

mean kinetic energy

$$E = \int_{-\infty}^{\infty} P(v) m_0 c^2 \gamma(v) dv$$

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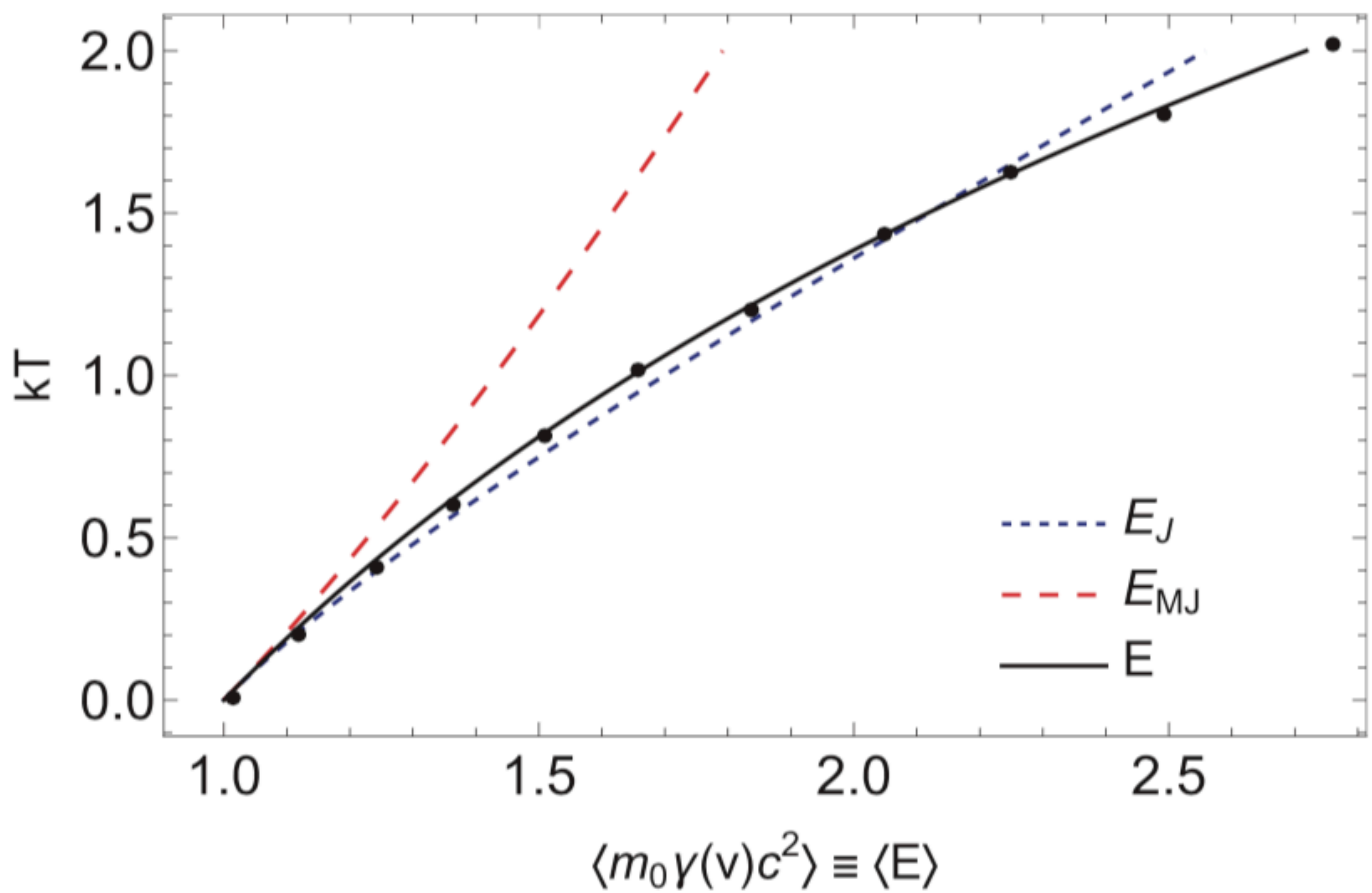

conclusions

- Jüttner is not LI
- derivation of a LI distribution of velocities based on the relativistic law of addition of velocities and CLT
- the LI distribution matches well with the numerical simulations
- temperature is LI

EMFC, F.T.L. Germani and I. Damião-Soares, Physica A 444 (2016) 963

EMFC, C.E. Cedeño, I. Damião-Soares, C. Tsallis, Chaos 32 (2022) 103110 thank you

"There are men who fight for a day and are good. There are men who fight for a year and they are better. There are men who fight for many years and are even better. But there are those who fight all their lives. These are the indispensable ones" (Bertold Brecht)

parabéns, Constantino!