Nonlinear dynamical systems: Time reversibility versus sensitivity to the initial conditions

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Statistical Mechanics for Complexity
—
A celebration of the 80th birthday of Constantino Tsallis

CBPF, November 6th, 2023

Previous collaborations with Constantino Tsallis on low dimensional dynamical systems

The quest for understanding the family of q indices

- q_{ent}: finite *entropy* production rate
- q_{sen}: q-exponential *sensitivity* of initial conditions
- q_{rel}: q-exponential rate of *relaxation* processes
- q_{stat}: q-gaussian *stationary* distribution

q_{sen}, q_{rel}

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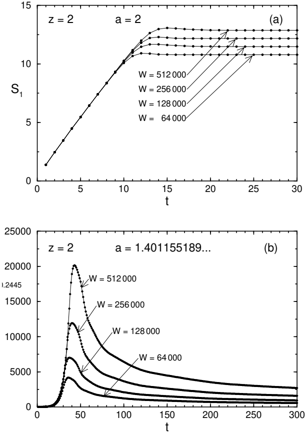
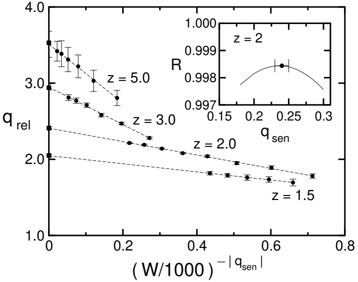
Nonequilibrium Probabilistic Dynamics of the Logistic Map at the Edge of Chaos

Ernesto P. Borges,[1,2] Constantino Tsallis,[1] Garín F. J. Añaños,[1,3] and Paulo Murilo C. de Oliveira[4]

$$x_{t+1} = 1 - a|x_t|^z \quad (z > 1; a \in [0,2]; x_t \in [-1,1])$$



- Sensitivity-based approach:
  $\Delta x(t) \propto \exp_{q_{sen}}(\lambda_{q_{sen}} t), \quad (q_{sen} \leq 1)$
- Relaxation-based approach:
  $W_{occ}(t) \propto \exp_{q_{rel}}(-t/\tau_{q_{rel}}), \quad (q_{rel} \geq 1)$



$q_{ent}, q_{sen}, q_{rel}, q_{stat}$: the standard map

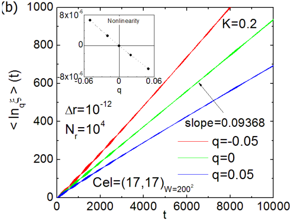
Statistical characterization of the standard map

J. Stat. Mech. (2017) 063403

Guiomar Ruiz[1,2], Ugur Tirnakli[2,3], Ernesto P Borges[4,5] and Constantino Tsallis[2,5,6,7]

$$p_{i+1} = p_i - K \sin x_i$$
$$x_{i+1} = x_i + p_{i+1}$$

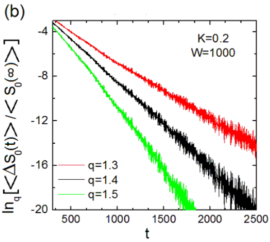
Generalized Pesin identity



Strong chaos: $q_{ent} = q_{sen} = 1$

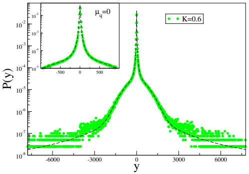
weak chaos: $q_{ent} = q_{sen} = 0$

Rate of relaxation of $S_{q_{rel}}$



$$\Delta S_{q_{ent}} \propto \exp_{q_{rel}}(-t/\tau_q)$$
$$q_{rel} = 1.4$$

Stationary distributions



$$P(y) = c_q \exp_{q_{stat}}\left(-\beta_q (y-\mu)^2\right) + c_1 \exp_1\left(-\beta_1 (y-\mu)^2\right)$$

$$q_{stat} = 1.935$$

q_{ent}, q_{sen}, q_{stat}: the web map

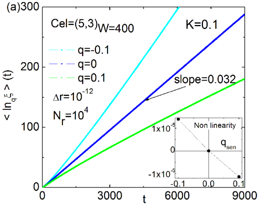
PHYSICAL REVIEW E 96, 042158 (2017)

Statistical characterization of discrete conservative systems: The web map

Guiomar Ruiz,[1,2,*] Ugur Tirnakli,[3,2,†] Ernesto P. Borges,[4,5,‡] and Constantino Tsallis[2,5,6,7,§]

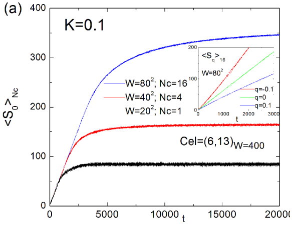
$$u_{i+1} = v_i$$
$$v_{i+1} = -u_i - K \sin v_i$$

Linear sensitivity to initial conditions



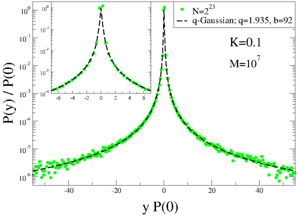
Strong chaos: $q_{ent} = q_{sen} = 1$
weak chaos: $q_{ent} = q_{sen} = 0$

Rate of entropy production



$q_{ent} = 0$

Stationary distributions



$$P(y) \propto \exp_{q_{stat}} \left(-\beta_q (y-\mu)^2 \right)$$
$$q_{stat} = 1.935$$

q_{ent} for multidimensional systems

Chaos, Solitons and Fractals 174 (2023) 113855

Entropy evolution at generic power-law edge of chaos

Constantino Tsallis[a,b,c,d], Ernesto P. Borges[b,e], Angel R. Plastino[f,*]

Entropy production rate

$$K_{q_{ent}} = \lim_{t\to\infty} \lim_{W\to\infty} \lim_{M\to\infty} \frac{\langle S_q(t)\rangle}{t} = \begin{cases} \infty, & q < q_{ent} \\ \text{finite}, & q = q_{ent} \\ 0, & q > q_{ent} \end{cases}$$

D: Number of positive q-generalized Lyapunov coefficients

Conjecture: $$\frac{1}{1-q_{ent}} = \sum_{k=1}^{D} \frac{1}{1-q_{ent,k}}$$ for fully independent dynamical variables

$$x_{t+1} = f_{a,\zeta}(x_t) \equiv 1 - a|x_t|^{\zeta} \quad (\zeta > 1; a \in [0,2]; x_t \in [-1,1])$$

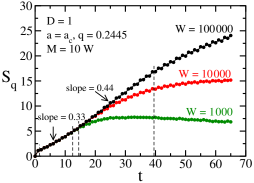
- $D = 1$: $x_{t+1} = f_{a_1,\zeta_1}(x_t)$
- $D = 2$: $[x_{t+1}, y_{t+1}] = [f_{a_1,\zeta_1}(x_t), f_{a_2,\zeta_2}(y_t)]$,

Verification of the conjecture $\frac{1}{1-q_{ent}} = \sum_{k=1}^{D} \frac{1}{1-q_{ent,k}}$ for the ζ-logistic map

$$x_{t+1} = f_{a,\zeta}(x_t) \equiv 1 - a|x_t|^\zeta \quad (\zeta > 1; a \in [0,2]; x_t \in [-1,1])$$

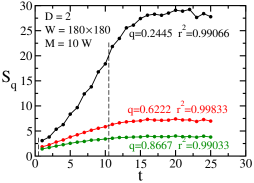
$\zeta = 2$

$D = 1$



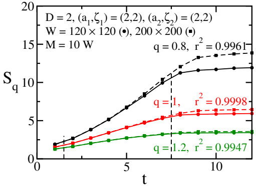
$D = 2$

$$q_{ent} = 0.6222 \approx 1 - \frac{1-0.2445}{2}$$



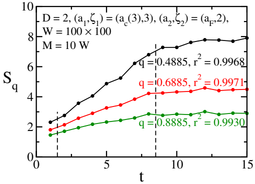
$D = 2$

$\zeta_1 = 2; \zeta_2 = 2$



$\zeta_1 = 3; \zeta_2 = 2$

$$\frac{1}{1-q_{ent}} = \frac{1}{1-0.6885} \approx \frac{1}{1-0.4720} + \frac{1}{1-0.2445}$$

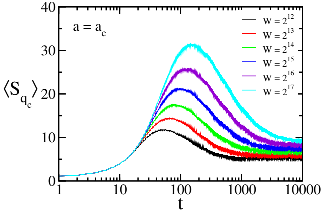


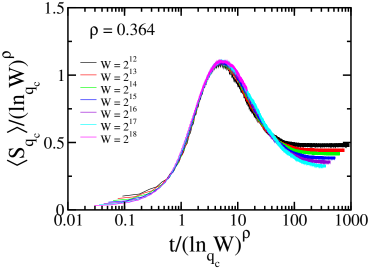
Logistic map: overshooting and stationary value of S_{q_e}

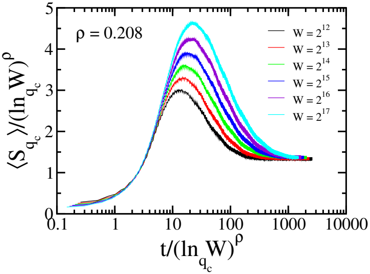
Chaos, Solitons and Fractals 171 (2023) 113431

Time evolution of nonadditive entropies: The logistic map

Constantino Tsallis [a,b,c,d], Ernesto P. Borges [e,b,*]







Symmetrization of a time series

The method splits a given time series

$$\{x_t\}, \quad t = 0, \ldots, t_f$$

into two series with regard to the time reversal symmetry $t \leftrightarrow (t - t_f)$

Symmetric Series

$$S_t \equiv \frac{x_t + x_{t_f - t}}{2}$$

Antisymmetric Series

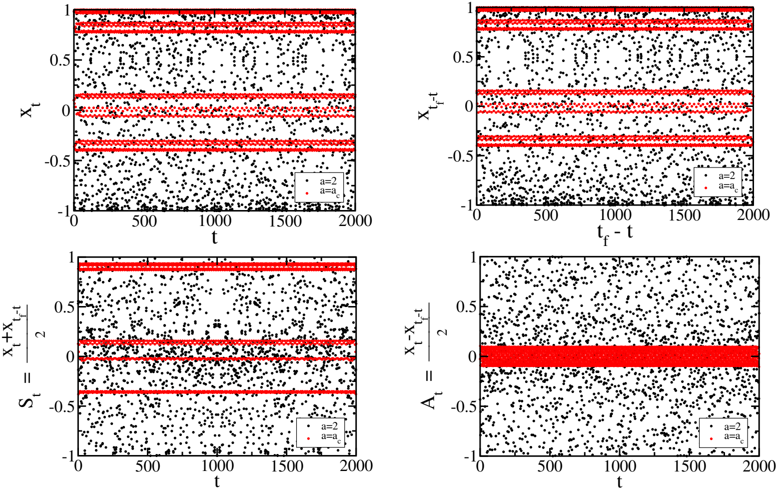
$$A_t \equiv \frac{x_t - x_{t_f - t}}{2}$$

$$x_t = S_t + A_t$$

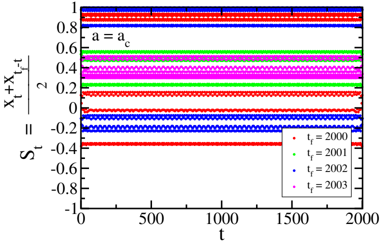
An instance: the logistic map $x_{t+1} = 1 - a x_t^2$

$$x_0 = 0.1, \quad t_f = 2000$$

black dots: $a = 2$, red dots: $a = a_c = 1.401155189092 05\ldots$

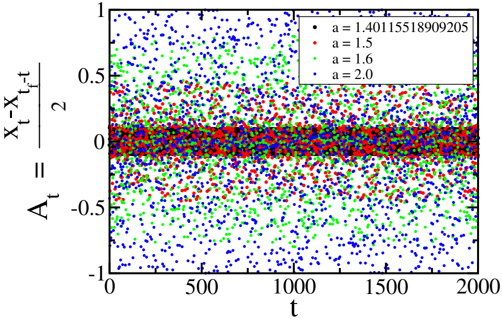


Effect of the final time



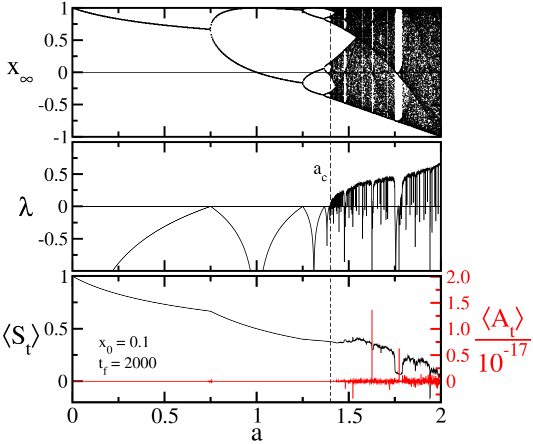
Influence of the control parameter a on A_t

$$(x_0, t_f) = (0.1, 2000)$$

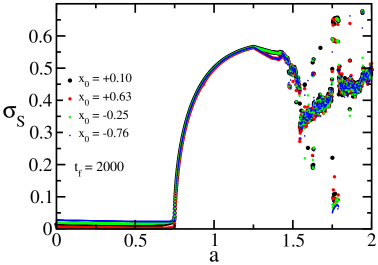


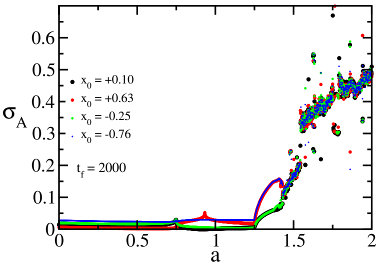
Influence of the control parameter a on $\langle S_t \rangle$ and $\langle A_t \rangle$

$$(x_0, t_f) = (0.1, 2000)$$

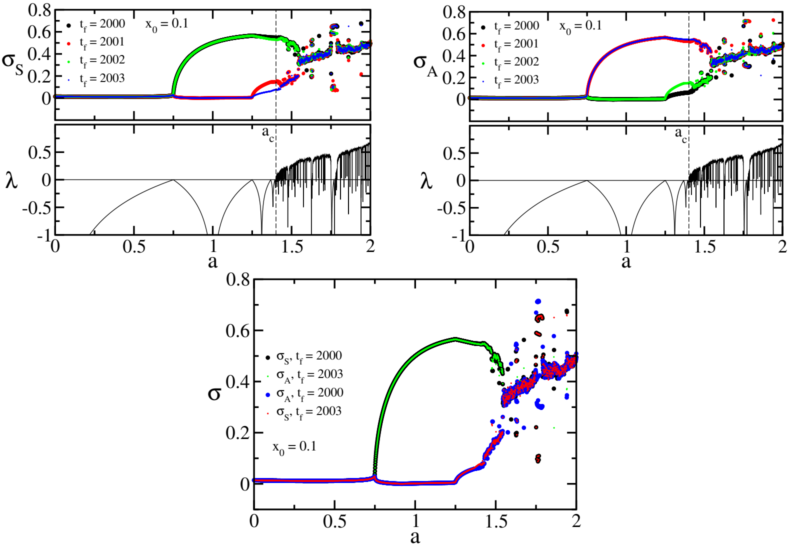


Influence of the initial condition on the standard deviations σ_S and σ_A for fixed t_f

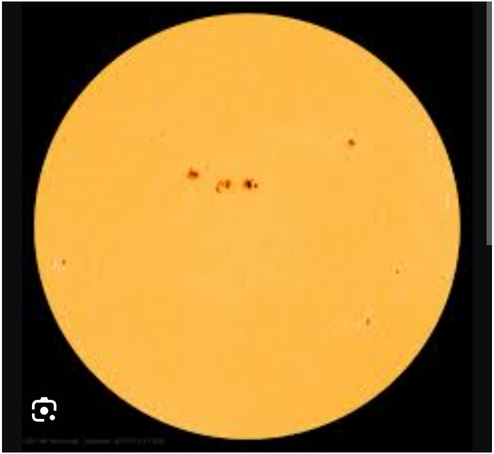




Influence of t_f on (σ_S, σ_A)

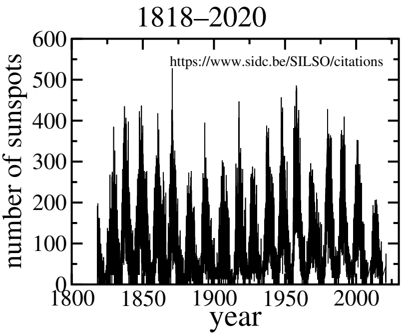


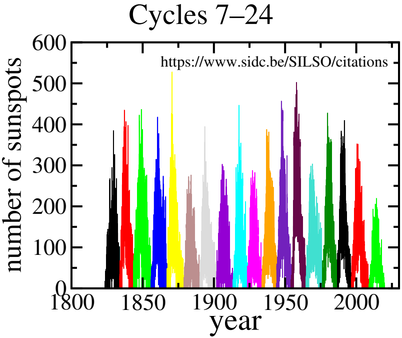
Another instance: Sunspots



http://iau.org

Sunspots Cycles

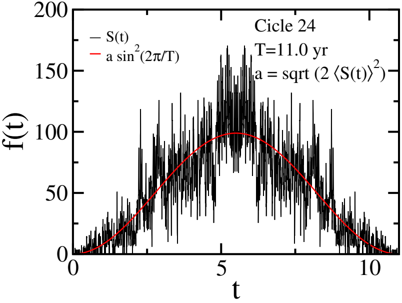




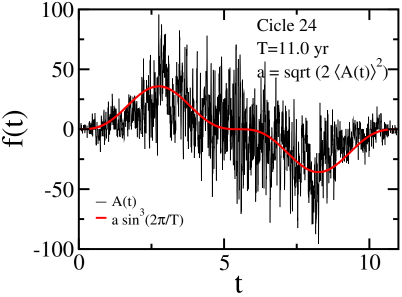
Cycle 24

Dec 16, 2009 – Dec 16, 2019
Period: 11.0 years

$S(t)$



$A(t)$



Happy Birthday, Constantino!