Celia Anteneodo
Puc-Rio

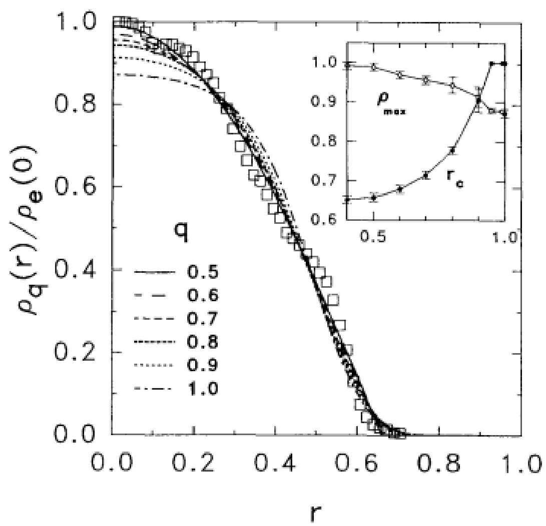
Turbulence

ELSEVIER
Journal of Molecular Liquids
Volume 71, Issues 2–3, April 1997, Pages 255-267

Two-dimensional turbulence in pure-electron plasma: A nonextensive thermostatistical description

Celia Anteneodo, Constantino Tsallis

Huang and Driscoll (1994) studied the two-dimensional turbulent metaequilibrium state that appears in an experiment in which pure-electron plasma evolves in the interior of a conducting cylinder (of radius R_w) in the presence of an external axial magnetic field.
Also related to work by B.M. Boghosian.



Density profiles of metaequilibrum state

https://www.sciencedirect.com/science/article/pii/S0167732297000160

Long-range interactions

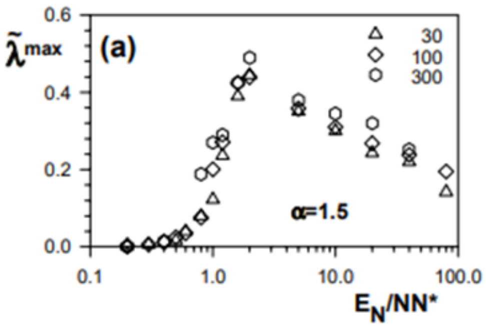
VOLUME 80, NUMBER 24 PHYSICAL REVIEW LETTERS 15 JUNE 1998

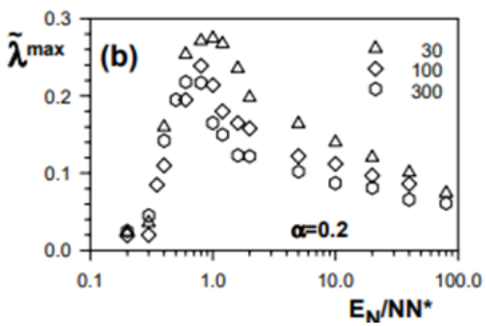
Breakdown of Exponential Sensitivity to Initial Conditions: Role of the Range of Interactions

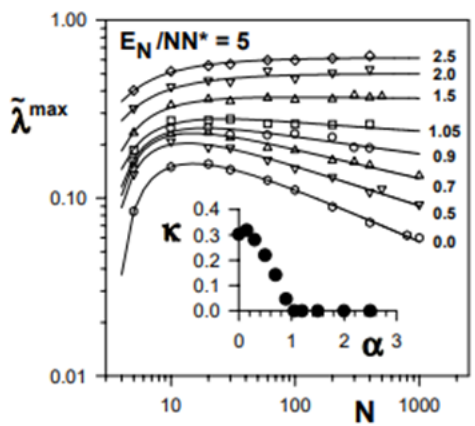
Celia Anteneodo[1] and Constantino Tsallis[2]

Let us consider the following $d = 1$ classical Hamiltonian (with periodic boundary conditions):

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} L_i^2 + \frac{1}{2}\sum_{i \neq j} \frac{1 - \cos(\theta_i - \theta_j)}{r_{ij}^{\alpha}} = E_k + E_p \quad (\alpha \geq 0;\, r_{ij} = 1, 2, 3, \dots), \qquad (1)$$







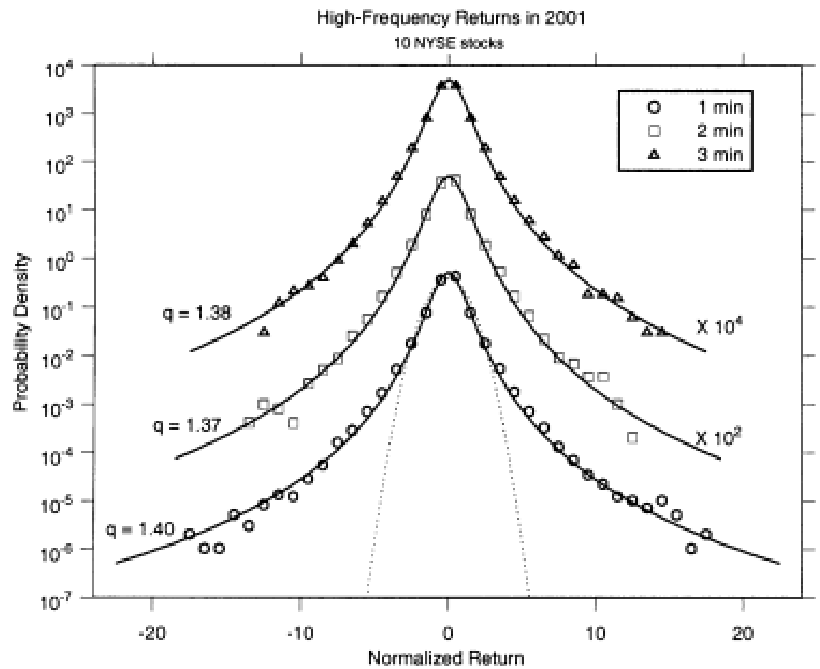
https://journals.aps.org/prl/pdf/10.1103/PhysRevLett.80.5313

Econophysics

Physica A: Statistical Mechanics and its Applications
Volume 324, Issues 1–2, 1 June 2003, Pages 89-100

Nonextensive statistical mechanics and economics

Constantino Tsallis [a], Celia Anteneodo [a], Lisa Borland [b], Roberto Osorio [b]



Empirical distributions (points) and q-Gaussians (solid lines) for normalized returns

https://www.sciencedirect.com/science/article/pii/S0378437103000426

Econophysics

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

Risk aversion in economic transactions

C. Anteneodo[1], C. Tsallis[1] and A. S. Martinez[2]

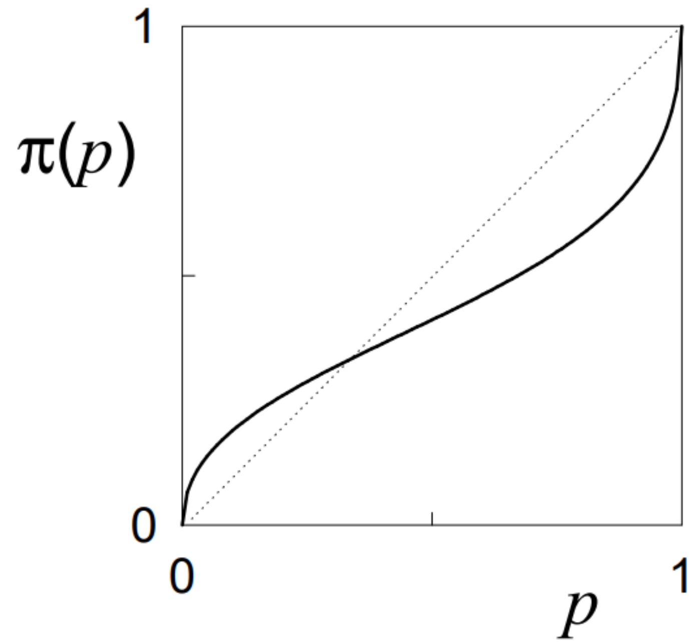
2002 EDP Sciences

Europhysics Letters, Volume 59, Number 5

Based on generalization of expected utility theory" (EUT)

$$E(\mathcal{P}) \; = \; \sum_{i=1}^{n} \chi(x_i)\,\Pi(p_i),$$

Tversky & Kahneman (Nobel 2002)



https://www.sciencedirect.com/science/article/pii/S0378437103000426

Econophysics

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

Risk aversion in economic transactions

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2002 EDP Sciences

Europhysics Letters, Volume 59, Number 5

In conclusion, the type of conditions limiting indebtedness are critical for defining the nature of the long term evolution.
The details of this steady state depend, among other factors, on the distribution of the parameter q of the operators. One also observes that the final state is invariant under initial redistribution of money.
Paradoxically enough, some level of cheating avoids extreme wealth inequality to become the stationary state. However, one must keep in mind that the distribution of q is kept fixed along the dynamics and, therefore, the psychological effect of asset position is not taken into account in the present model. Such dynamics would provide an improved, more realistic model.

→ Constantino invited conference at the "International Public Seminar of the Year", 27 August 2002, Jakarta, Indonesia

Multiplicative noise

Multiplicative noise: A mechanism leading to nonextensive statistical mechanics

Celia Anteneodo; Constantino Tsallis

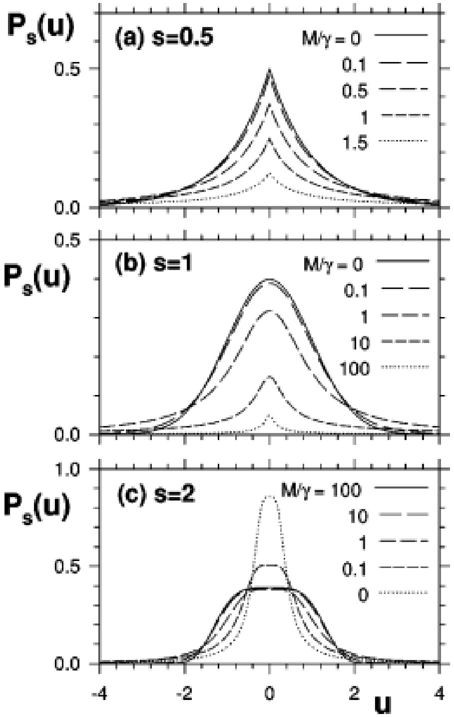
Check for updates

J. Math. Phys. 44, 5194–5203 (2003)
https://doi.org/10.1063/1.1617365 Article history

$$\dot{u} = f(u) + g(u)\xi(t) + \eta(t),$$

$$\langle \xi(t)\xi(t')\rangle = 2M\,\delta(t-t'), \quad \langle \eta(t)\eta(t')\rangle = 2A\,\delta(t-t'),$$

$$\rightarrow \quad P_s(u) \propto [1+(q-1)\beta[g(u)]^2]^{1/(1-q)},$$



https://pubs.aip.org/aip/jmp/article/44/11/5194/447749/Multiplicative-noise-A-mechanism-leading-to

Generalized entropy

Journal of Physics A: Mathematical and General

Maximum entropy approach to stretched exponential probability distributions

C Anteneodo[1] and A R Plastino[2]

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Journal of Physics A: Mathematical and General, Volume 32, Number 7

$$S_\eta = \sum_{i=1}^{w} s_\eta(p_i)$$

where $$s_\eta(p_i) \equiv \Gamma\left(\frac{\eta+1}{\eta}, -\ln p_i\right) - p_i \Gamma\left(\frac{\eta+1}{\eta}\right)$$

→ stretched exponentials

Here, η is a positive real number,

$$\Gamma(\mu, t) = \int_t^\infty y^{\mu-1} e^{-y}\, dy = \int_0^{\exp(-t)} [-\ln x]^{\mu-1}\, dx \qquad \mu > 0$$

is the complementary incomplete Gamma function, and $\Gamma(\mu) = \Gamma(\mu, 0)$ the Gamma function.

Generalized entropy

Journal of Physics A: Mathematical and General

Maximum entropy approach to stretched exponential probability distributions

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→ stretched exponentials

- Kinchin axioms,
- Positivity
- Concavity
- Certainty
- Equiprobability
- Nonextensivity
- Jaynes thermodynamic relations

https://web.archive.org/web/20050301070437id_/http://www.cbpf.br:80/~celia/paper14.pdf

Constantino,

happy birthday!

congratulations!

and thank you!

EQUILIBRATE OR NOT TO EQUILIBRATE THAT IS THE QUESTION

QUASI-EQUILIBRIUM STATES IN NONCONFINING FIELDS

C Anteneodo, M dos Santos - PUC-Rio, Brazil
E Barkai, D Kessler, L Defaveri - Bar-Ilan U, Israel

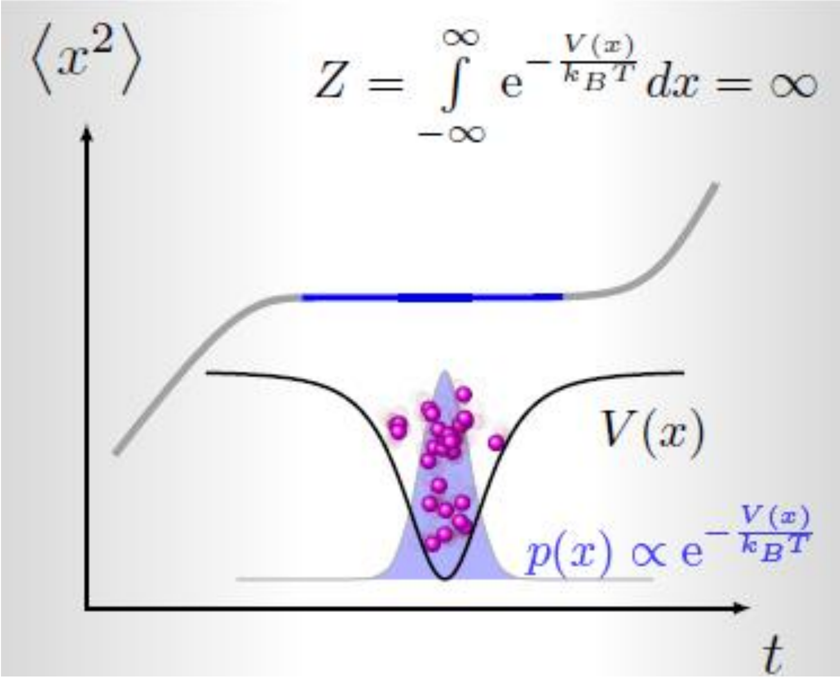
PUC RIO

Complex Systems PUC-Rio

inct-sc

Outline

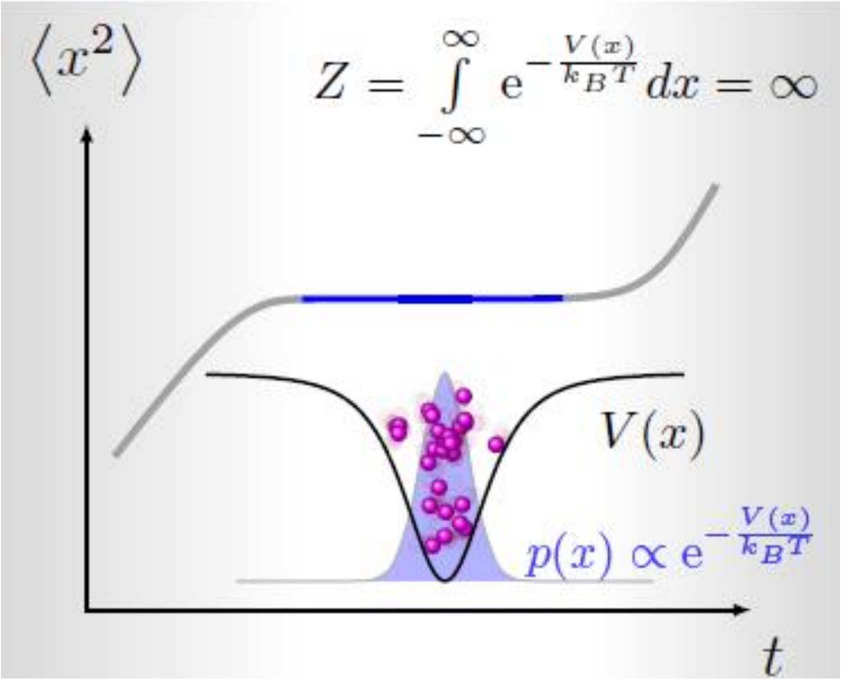
To equilibrate or not to equilibrate,
that is the question.



Regularized Boltzmann statistics
is the answer.

Outline

To equilibrate or not to equilibrate,
that is the question.

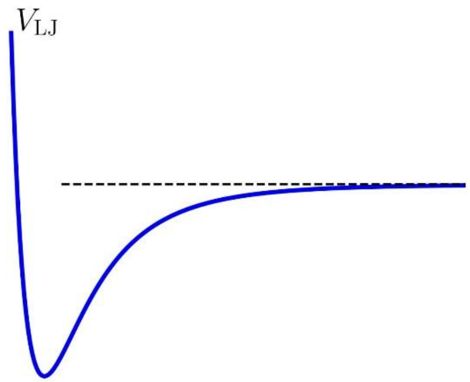


Regularized Boltzmann statistics
is the answer.

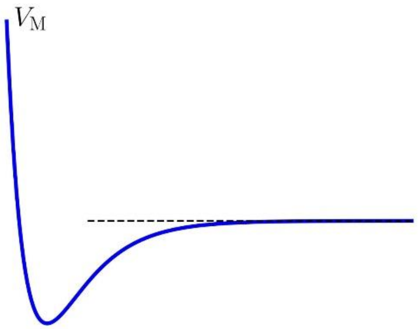
- Non-confinement
- Langevin
- Fokker-Planck
- Quasi-equilibrium
- Regularization
- Bounded domain
- Eigenfunctions
- Fractional
- Final remarks

Non-confinement

Non-confining potentials are ubiquitous in Nature



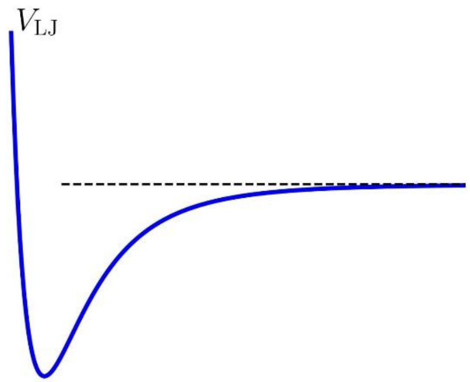
$$V_{LJ} = V_m \left(\frac{\sigma^{12}}{r^{12}} - \frac{\sigma^{6}}{r^{6}} \right)$$

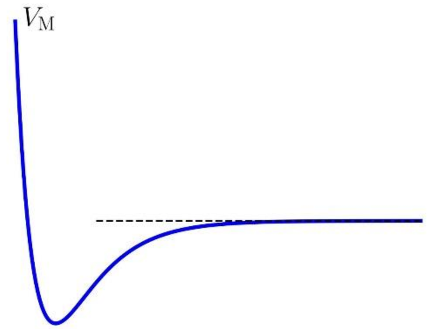


$$V_M = V_m \left(1 - e^{-a(r-r_e)} \right)^2$$

Non-confinement

Non-confining potentials are ubiquitous in Nature





$$V_{LJ} = V_m \left(\frac{\sigma^{12}}{r^{12}} - \frac{\sigma^6}{r^6} \right)$$

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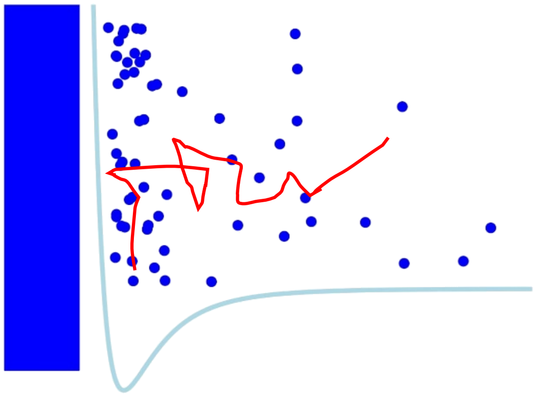
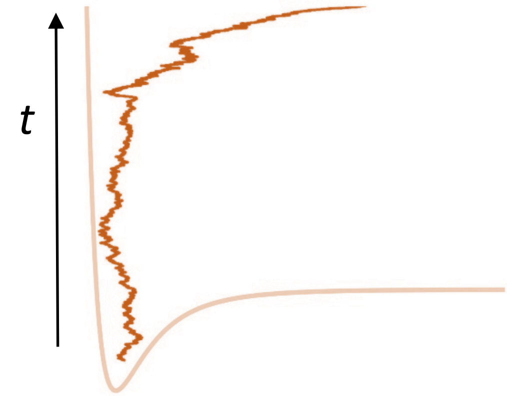




Non-confinement

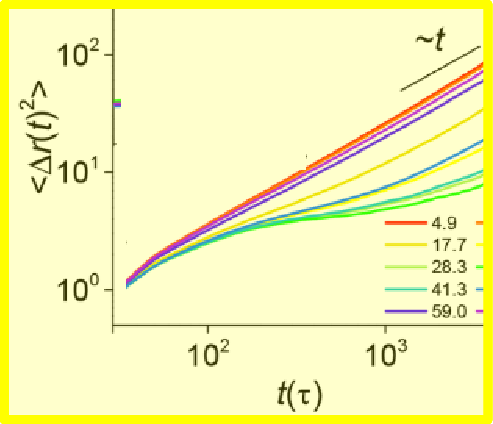
Some experiments where transient stagnation emerges

- Particles diffusing in heterogeneous media
  Liu *et al.*, ACS Nano 16 (2022)

- Probe particles in micellar solutions
  Bellour et al., Eur. Phys. J. E 8 (2002)
  Galvan-Miyoshi et al., Eur. Phys. J. E 26 (2008)
  Jeon *et al.*, New J. Phys. 15 (2013)

- Nanoparticles in semi-flexible networks
  Xu *et al.*, ACS Nano 15, (2021)

- Excitons in semiconductors
  Kurilovich *et al.*, Phys Ch Ch Phys 22 (2020), 24 (2022)



PRE 108, 024133 (2023)
Entropy 23, 131 (2021)
PRR 2, 043088 (2020)

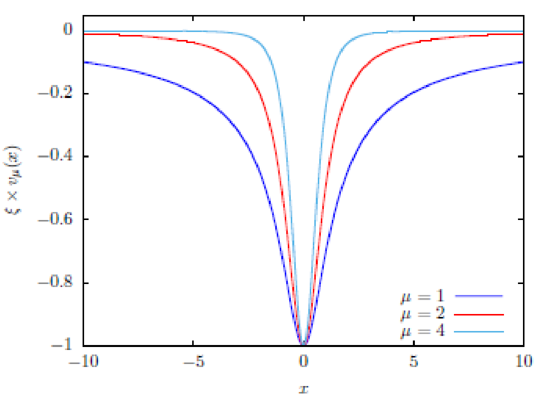
Langevin dynamics

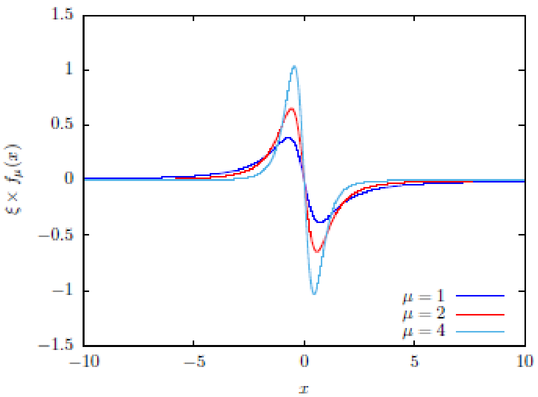
overdamped dynamics of a Brownian particle in 1D

$$\gamma \frac{dx}{dt} = F_\mu(x) + \sqrt{2\gamma k_B T}\,\eta(t)\,,$$

$\eta(t)$ is a zero-mean Gaussian white noise with $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$, and $F_\mu(x)$ is derived from the (asymptotically flat) potential

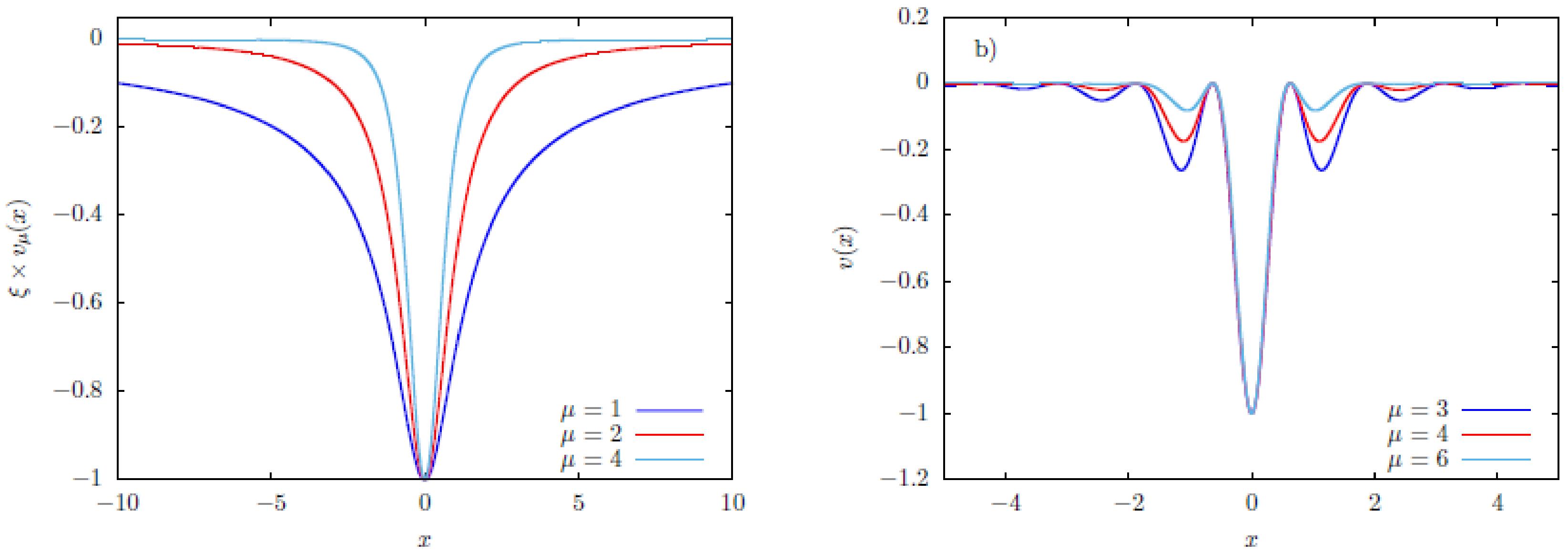
$$V_\mu(x) = -\frac{U_0}{(1+(x/x_0)^2)^{\frac{\mu}{2}}} \qquad F_\mu(x) = -V'_\mu$$





Langevin dynamics

asymptotically flat potentials



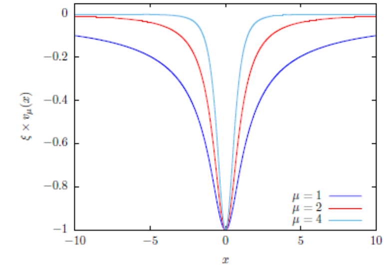
Fokker-Planck approach

for the overdamped dynamics of a Brownian particle in 1D

$$\frac{\partial}{\partial t}P(x,t) = D\left\{\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F_\mu(x)}{k_B T}\right\}P(x,t),$$

where $F_\mu(x)$ is related to the (asymptotically flat) potential

$$V_\mu(x) = -\frac{U_0}{(1+(x/x_0)^2)^{\frac{\mu}{2}}}.$$



For simplicity, we scale the variables as

$$\frac{x}{x_0} \to x\ , \quad -\frac{Dt}{x_0^2} \to t\ , \quad \frac{V_\mu(x)}{U_0} \to v_\mu(x)\ , \quad \frac{k_B T}{U_0} \to \xi\ ,$$

leading to the reduced FP equation

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial^2}{\partial x^2}P(x,t) + \frac{1}{\xi}\frac{\partial}{\partial x}\left\{\frac{\partial v_\mu(x)}{\partial x}P(x,t)\right\}$$

Quasi-equilibrium

If there is a stationary solution, then it must obey

$$\frac{\partial^2}{\partial x^2} P(x,t) + \frac{1}{\xi}\frac{\partial}{\partial x}\left\{\frac{\partial v_\mu(x)}{\partial x} P(x,t)\right\} \quad = \quad 0,$$

which gives the Boltzmann-Gibbs PDF:

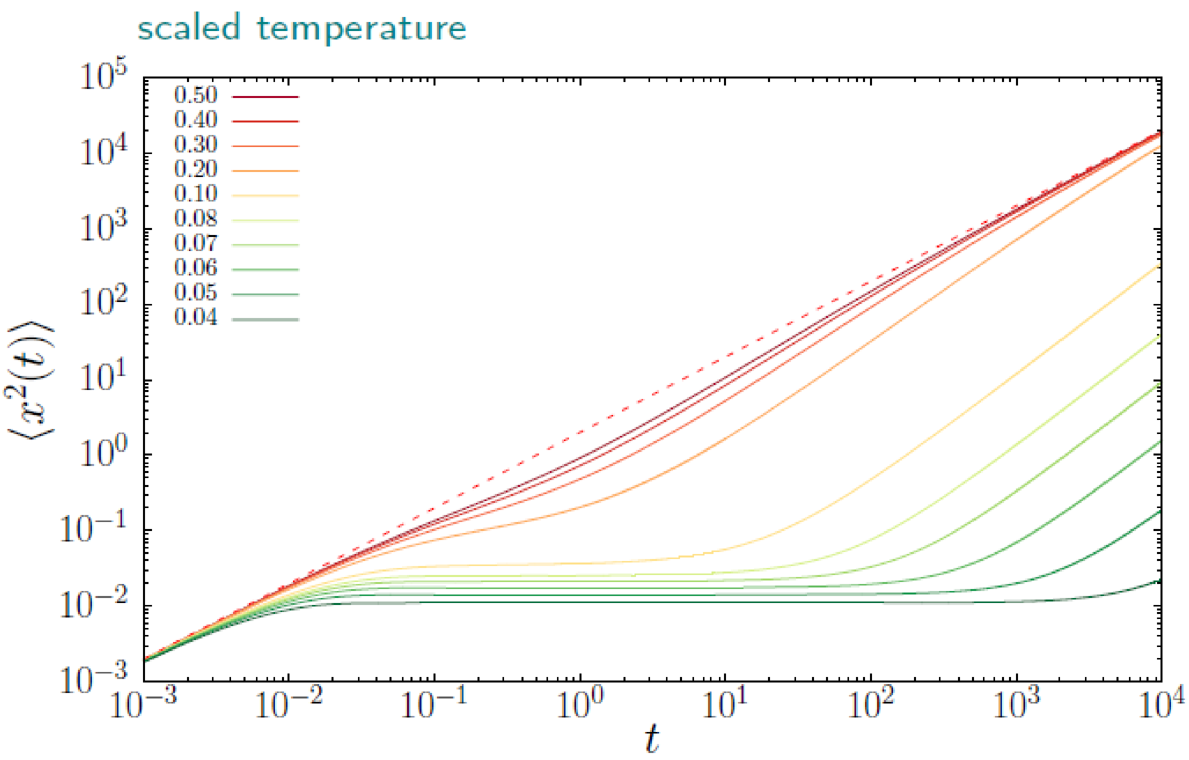
$$P(x) \sim e^{-\frac{v_\mu(x)}{\xi}}.$$

But due to the flatness of the potential, such a distribution is non-normalizable

$$Z = \int_{-\infty}^{\infty} e^{-\frac{v_\mu(x)}{\xi}} dx = \infty.$$

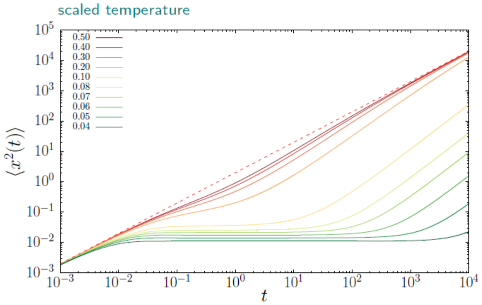
Quasi-equilibrium

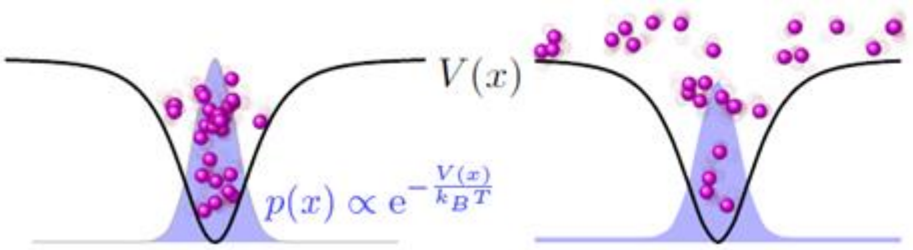
Numerically solving the FPE and computing the MSD



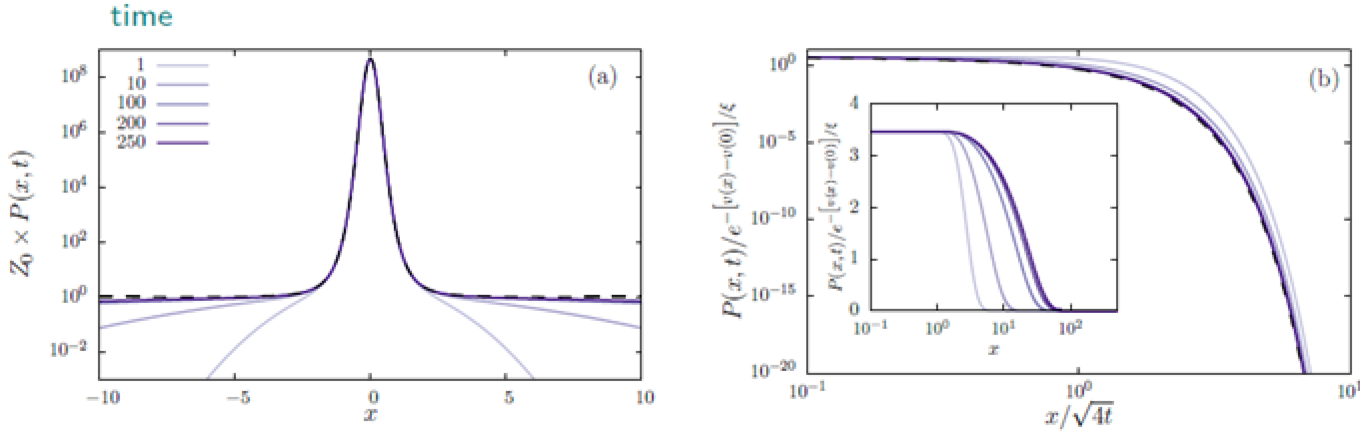
$$v_4(x) = -1/(1+x^2)^2$$

Quasi-equilibrium





– At early times $P(x,t)$ expands quickly to fill the bottom of the well.
– At intermediate times $P(x,t)$ remains *almost* stationary for small x while continues expanding for large x.
– At very long times, free diffusion.

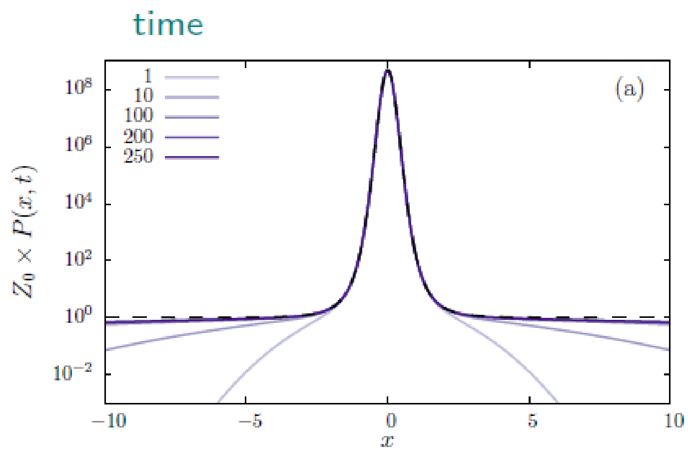


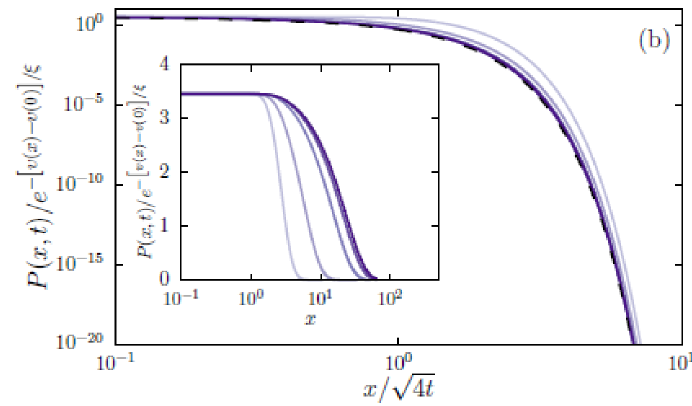
Regularization

First, we consider an approximate solution
(valid for intermediate timescales)

For small x: $P(x,t) \sim e^{-\frac{v_\mu(x)}{\xi}}$

For large x: $P(x,t) \sim \mathrm{erfc}\left(\frac{x}{\sqrt{4t}}\right)$





We can match both solutions to obtain

$$P(x,t) \simeq C\, e^{-v_\mu(x)/\xi}\, \mathrm{erfc}(x/\sqrt{4t})$$

Phys Rev Research 2, 043088 (2020)

Expansion in eigenfunctions

To obtain a more formal and complete description of the PDF at intermediate times, we use an expansion in the eigenfunctions of the Fokker-Planck operator.

$$P(x,t) = e^{v(0)/\xi}\left\{\mathcal{I}(x)N_0^{-1} + \sum_{\{k\}} N_k^{-1}\Psi_k(0)\Psi_k(x)e^{-k^2 t}\right\},$$

The eigenvalue band is continuous with a large contribution arising from values near the zeroth eigenmode.

We find explicitly an approximate result equivalent to our heuristic form

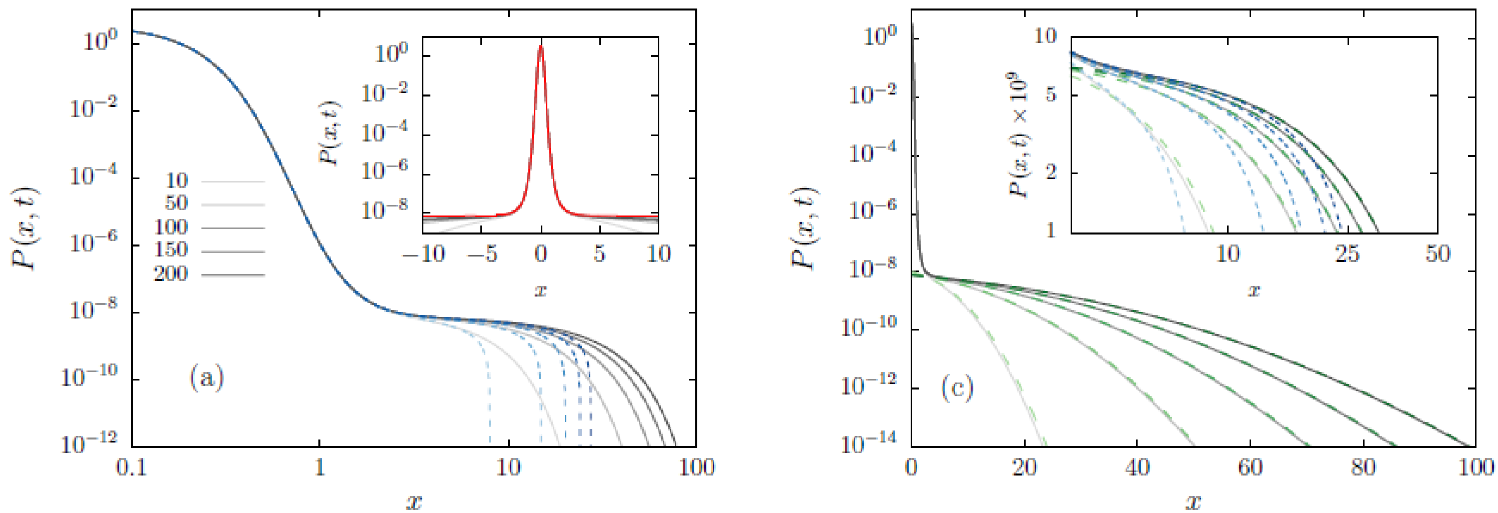
$$P(x,t) \simeq C\, e^{-v_\mu(x)/\xi}\mathrm{erfc}(x/\sqrt{4t})$$

Entropy 23, 131 (2021)

Expansion in eigenfunctions

$$P^{\mathrm{I}}(x,t) = \frac{e^{-v(x)/\xi}}{2Z_0}\left(1 - \frac{1}{\sqrt{\pi t}}\frac{g^{\mathrm{I}}(x)}{Z_0}\right)$$

$$P^{\mathrm{III}}(x,t) = \frac{1}{2Z_0}\,\mathrm{erfc}\left(\frac{x-\phi}{2\sqrt{t}}\right).$$



This method is very powerful, as it allows us to calculate corrections to the almost time-independent solution.

Entropy 23, 131 (2021)

Regularization

$$P(x,t) \simeq C\, e^{-v_\mu(x)/\xi}\, \mathrm{erfc}(x/\sqrt{4t})$$

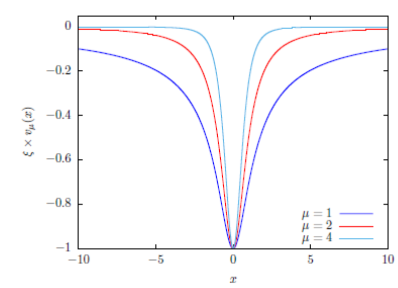
The normalization constant can be evaluated by reorganizing terms

$$\begin{aligned}
\frac{1}{2C} &\simeq \int_0^\ell e^{-v(x)/\xi} dx + \int_\ell^\infty \mathrm{erfc}\left(\frac{x}{\sqrt{4t}}\right) dx \\
&\simeq \underbrace{\int_0^\infty (e^{-v(x)/\xi} - 1) dx}_{Z_0/2} - \underbrace{\int_\ell^\infty (e^{-v(x)/\xi} - 1) dx}_{R} + \int_0^\infty \mathrm{erfc}\left(\frac{x}{\sqrt{4t}}\right) dx \\
&\simeq Z_0/2 + \mathcal{O}(\sqrt{t})
\end{aligned}$$

$$V_\mu(x) = -\frac{U_0}{(1+(x/x_0)^2)^{\frac{\mu}{2}}}.$$

Leading to the regularized partition function ($\mu > 1$)

$$Z_0 = \int_0^\infty (e^{-v(x)/\xi} - 1) dx.$$

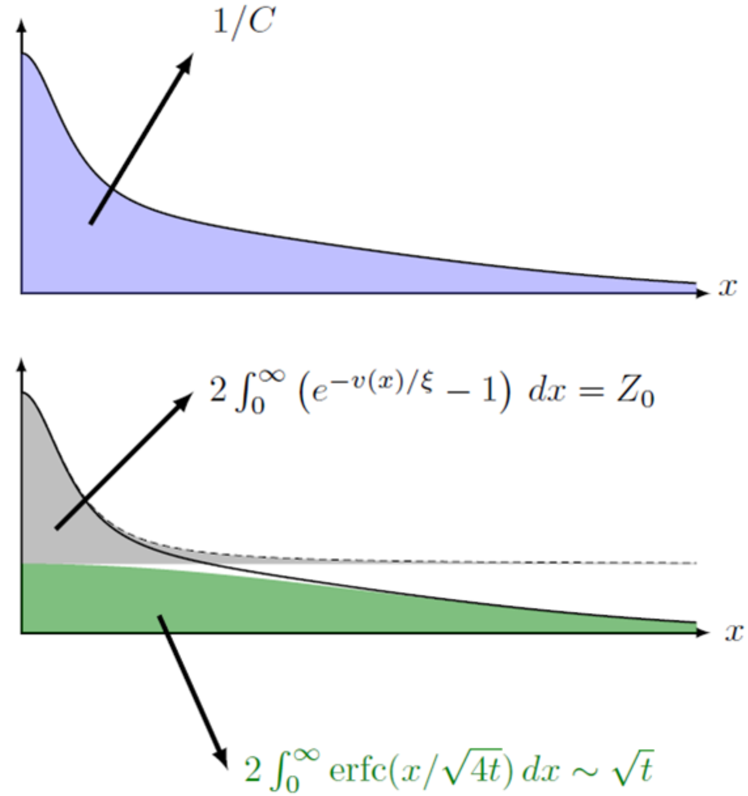


Regularization

$$\frac{1}{2C} \simeq \underbrace{\int_0^\infty (e^{-v(x)/\xi} - 1)dx}_{Z_0/2} - \underbrace{\int_\ell^\infty (e^{-v(x)/\xi} - 1)dx}_{R} + \int_0^\infty \mathrm{erfc}\left(\frac{x}{\sqrt{4t}}\right)dx$$

$$\simeq Z_0/2 + \mathcal{O}(\sqrt{t})$$

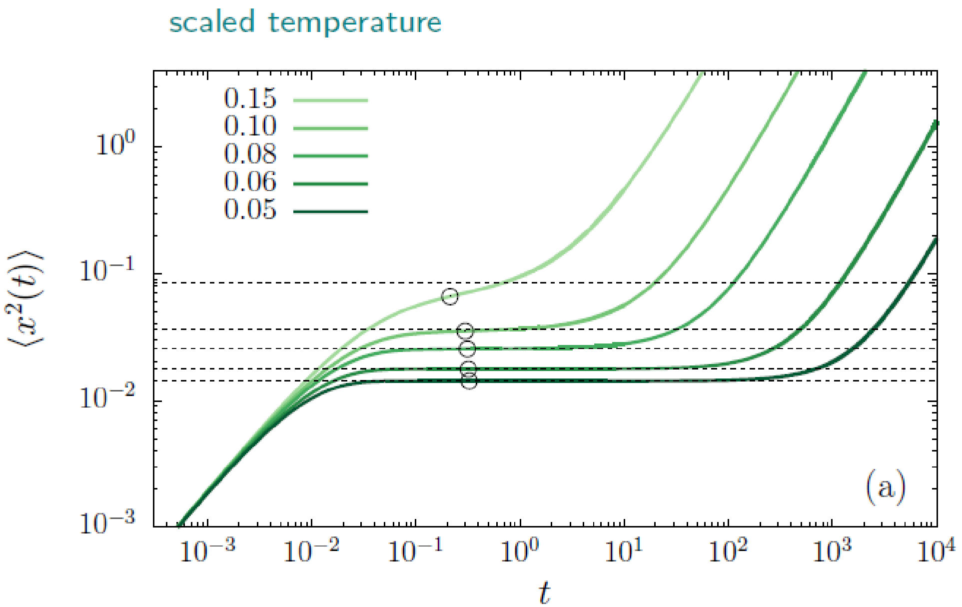
We can visualize the integration as



Regularization

Similarly, we can obtain the quasi-equilibrium MSD ($\mu > 3$)

$$\langle x^2 \rangle_{NQE} = \frac{\int_{-\infty}^{\infty} x^2 (e^{-v(x)/\xi} - 1) dx}{\int_{-\infty}^{\infty} (e^{-v(x)/\xi} - 1) dx}$$



$$v_4(x) = -1/(1+x^2)^2$$

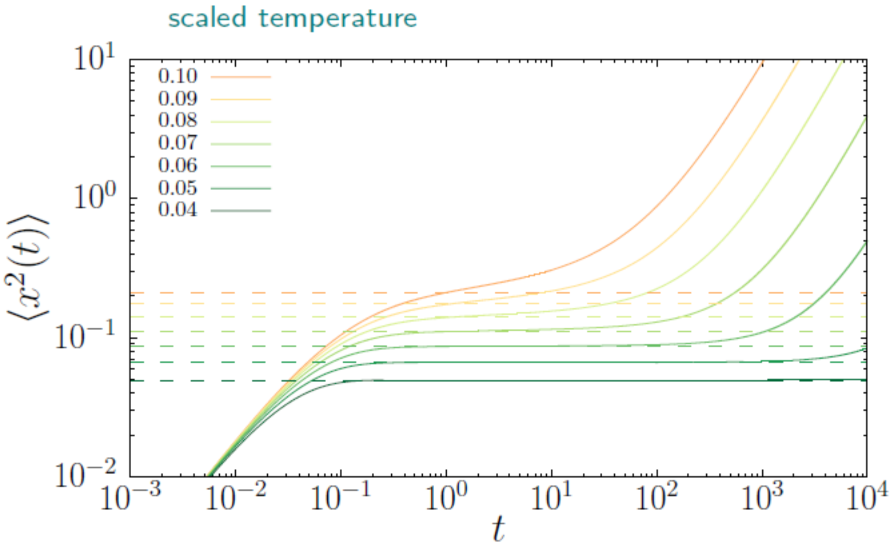
Regularization

We may need to subtract more terms of the exponential to ensure convergence, in other words:

$$Z_K = 2\int_0^\infty \left(e^{-v(x)/\xi} - \sigma_K(x;\xi)\right) dx\,, \qquad \sigma_K(x;\xi) \equiv \sum_{k=0}^{K} \left(-v(x)/\xi\right)^k / k!\,.$$

An example where we require more terms, $\mu = 1$ $[-v_1(x) \sim 1/x]$:

$$\langle x^2\rangle_{NQE} = \frac{\int_{-\infty}^{\infty} x^2 \left(e^{-v_1(x)/\xi} - 1 + \frac{v_1(x)}{\xi} - \frac{v_1^2(x)}{2\xi^2} + \frac{v_1^3(x)}{6\xi^3}\right) dx}{\int_{-\infty}^{\infty} \left(e^{-v_1(x)/\xi} - 1 + \frac{v_1(x)}{\xi}\right) dx}$$



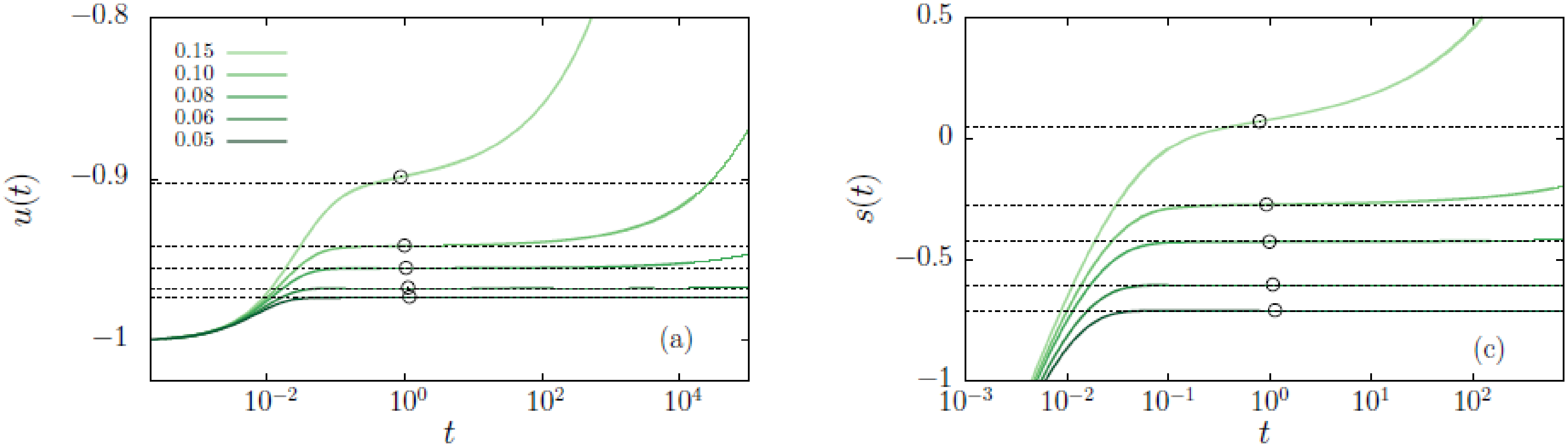
Regularization

Comparison of QE predictions with results obtained from numerical solution

$$u(t) \equiv E(t)/U_0 = \int_{-\infty}^{\infty} v(x) P(x,t) dx,$$

$$s(t) \equiv S(t)/k_B = -\int_{-\infty}^{\infty} \ln\left(P(x,t)\right) P(x,t) dx.$$

scaled temperature



$$v_4(x) = -1/(1+x^2)^2$$

Regularization

Using quasi-equilibrium it is possible to recover several relations within Boltzmann-Gibbs statistical mechanical formalism like

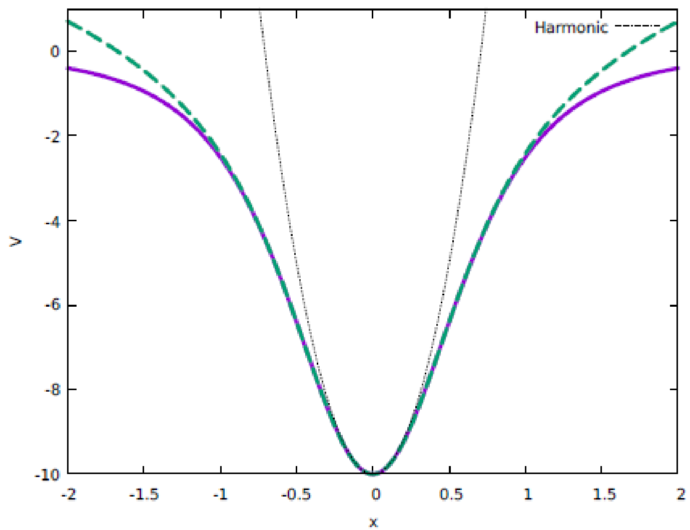
energy
$$u_{\rm QE} = \frac{\int_0^\infty v(x) e^{-v(x)/\xi} dx}{\int_0^\infty \left(e^{-v(x)/\xi} - 1\right) dx} \equiv \xi^2 \frac{\partial \ln Z_0}{\partial \xi}$$

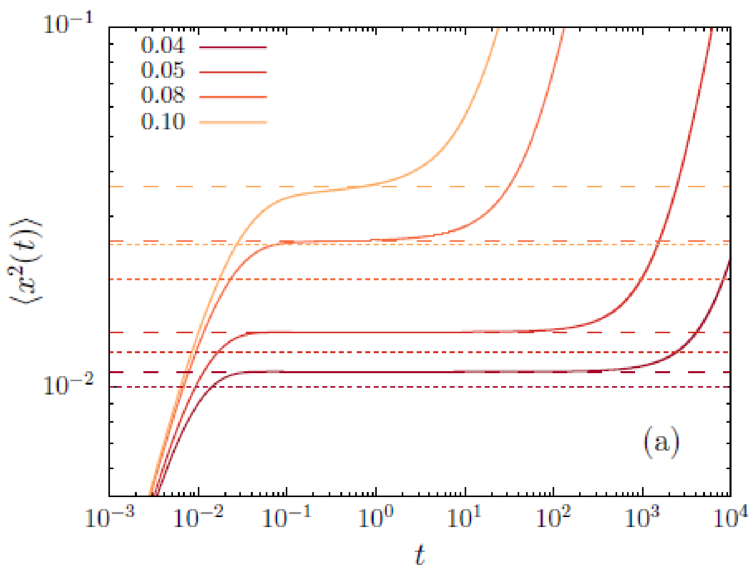
and entropy
$$s_{\rm QE} = u_{\rm QE}/\xi - \ln Z_0$$

Regularization

Our regularization corresponds to using the standard BG factor with the effective potential $v(x) \approx -\xi \log\left[e^{-V(x)/\xi} - 1\right]$

$\mu = 4,\ \xi = 0.1$

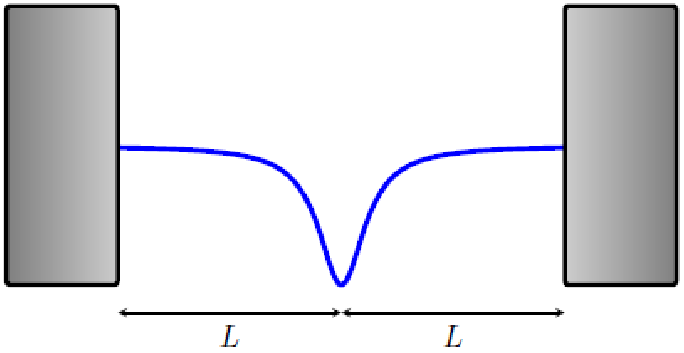




This effective potential works in a wider range than the simple low-temperature harmonic approx. for describing the potential at small x while still being confining.

Bounded domain

Alternatively, we can place the system in a box of size $2L$ (L is in the x_0 scale).
For intermediate box sizes, the partition function becomes L independent.



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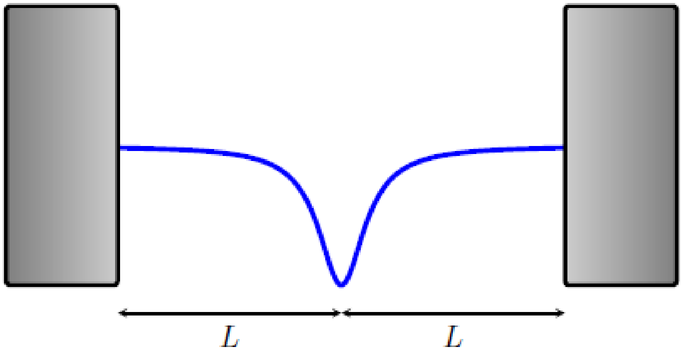
Bounded domain

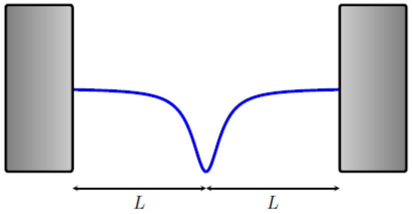
Alternatively, we can place the system in a box of size $2L$ (L is in the x_0 scale). For intermediate box sizes, the partition function becomes L independent.



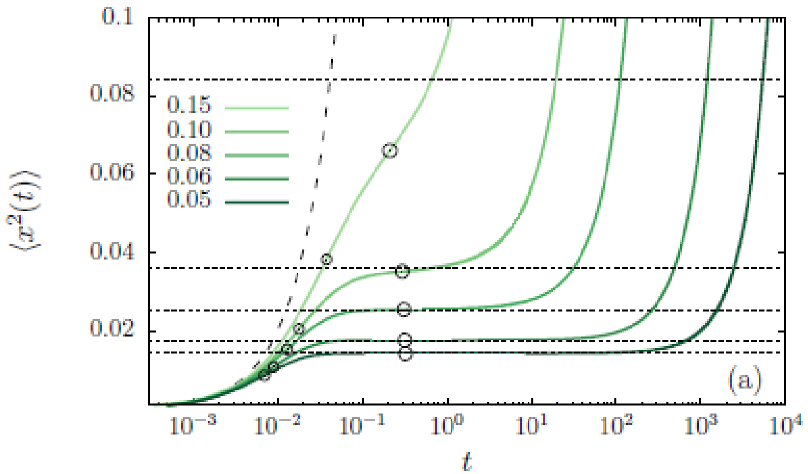
In such approach, for every L the system will be in *true* equilibrium allowing us to evaluate freely

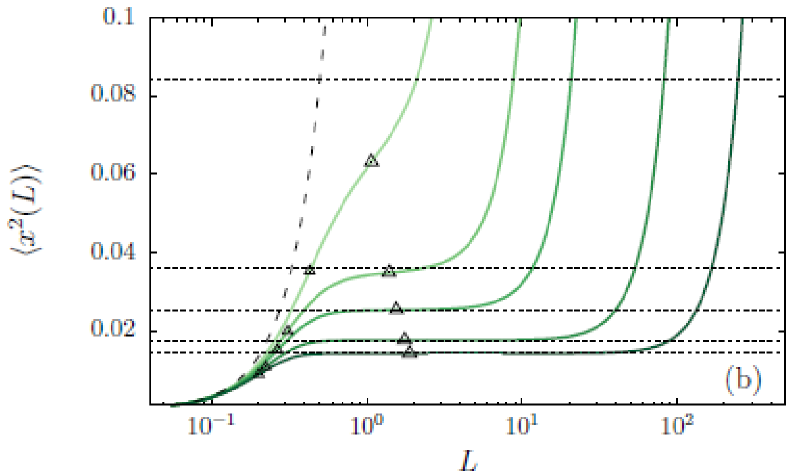
$Z(L) = 2\int_0^L e^{-v(x)/\xi} dx$, and $\qquad$ $P(x) = \frac{1}{Z(L)} e^{-v(x)/\xi}$, for $|x| < L$

Bounded domain



scaled temperature

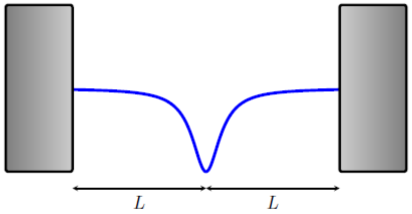




We can interpret that increasing *L* is equivalent to allowing the particle to keep diffusing further away from the well.

$$v_4(x) = -1/(1+x^2)^2$$

Bounded domain



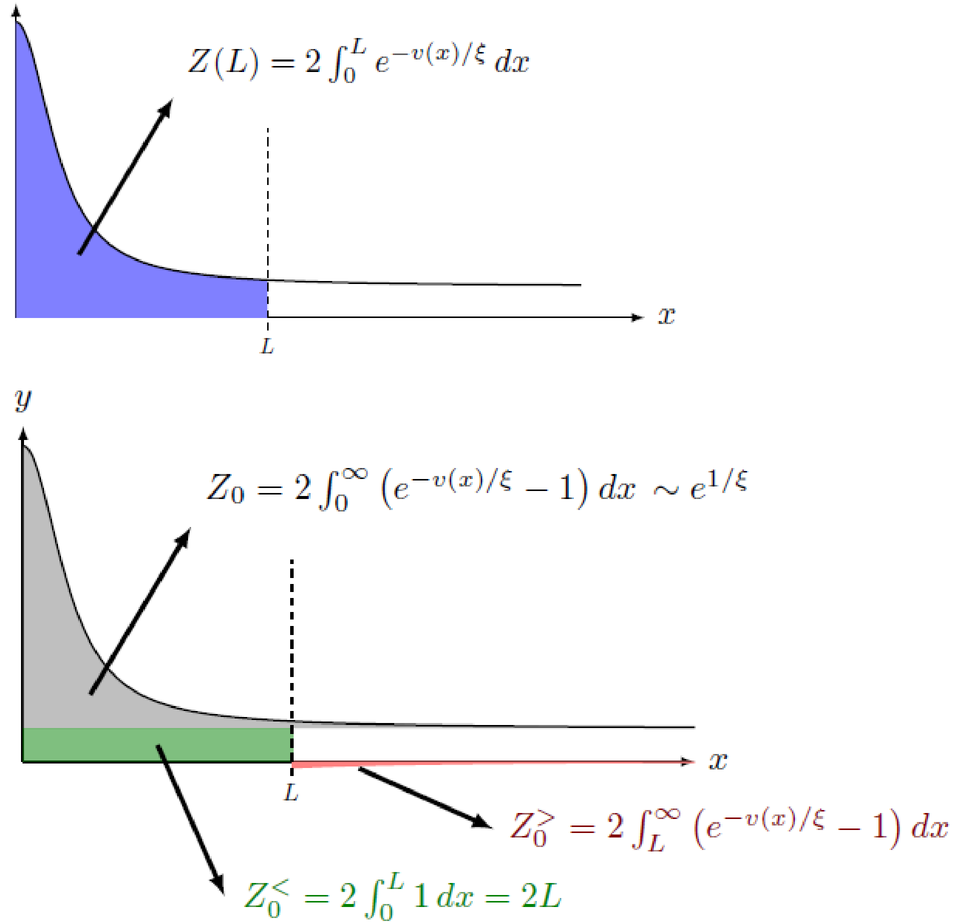
The approach to regularization is the same

$$Z(L) = 2\int_0^L \left(e^{-v(x)/\xi} - \sigma_K(x;\xi)\right)dx + 2\int_0^L \sigma_K(x;\xi)dx\,.$$

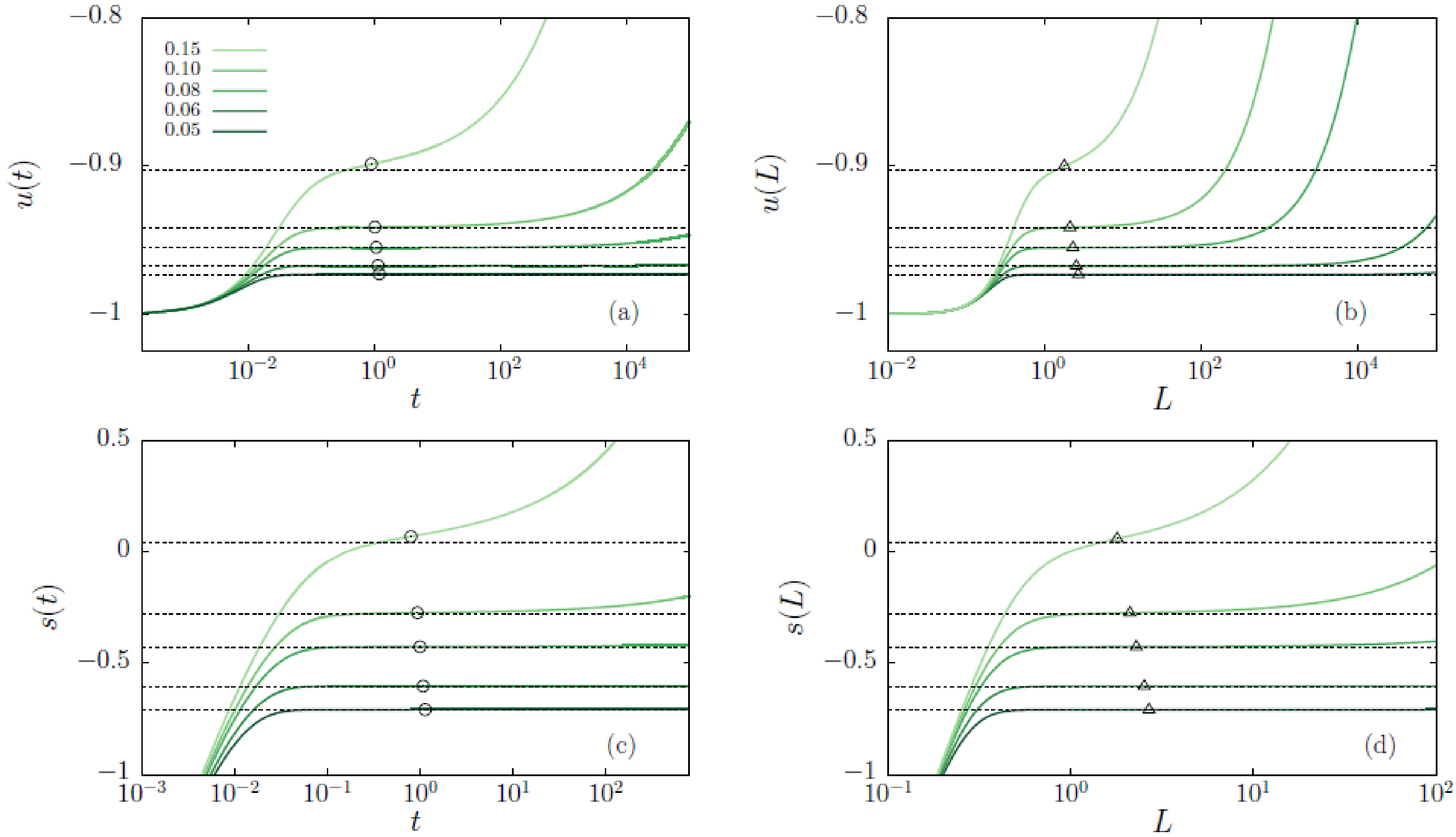
So if we define:

$$\begin{aligned} Z_K &= 2\int_0^\infty \left(e^{-v(x)/\xi} - \sigma_K(x;\xi)\right)dx \sim e^{1/\xi}\,, \\ Z_K^>(L) &= -2\int_L^\infty \left(e^{-v(x)/\xi} - \sigma_K(x;\xi)\right)dx \sim 0\,, \\ Z_K^<(L) &= 2\int_0^L \sigma_K(x;\xi)dx \sim 2L. \end{aligned}$$

We can visualize the integration as



scaled temperature



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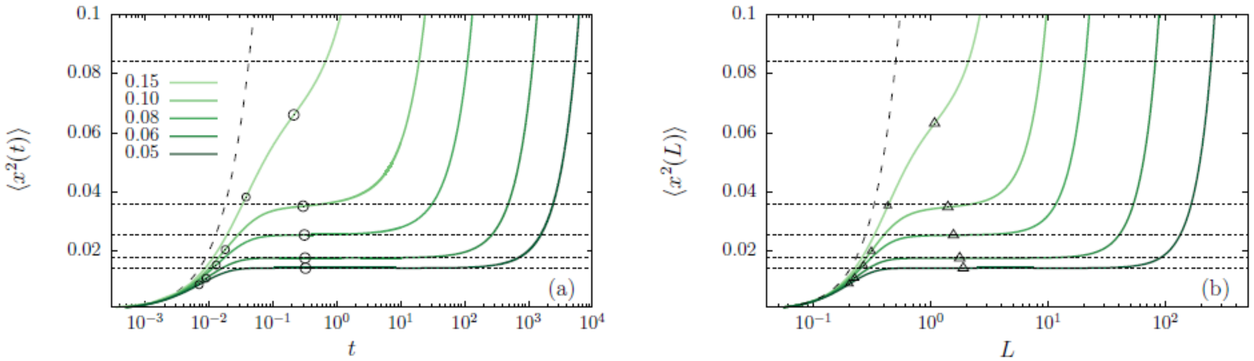
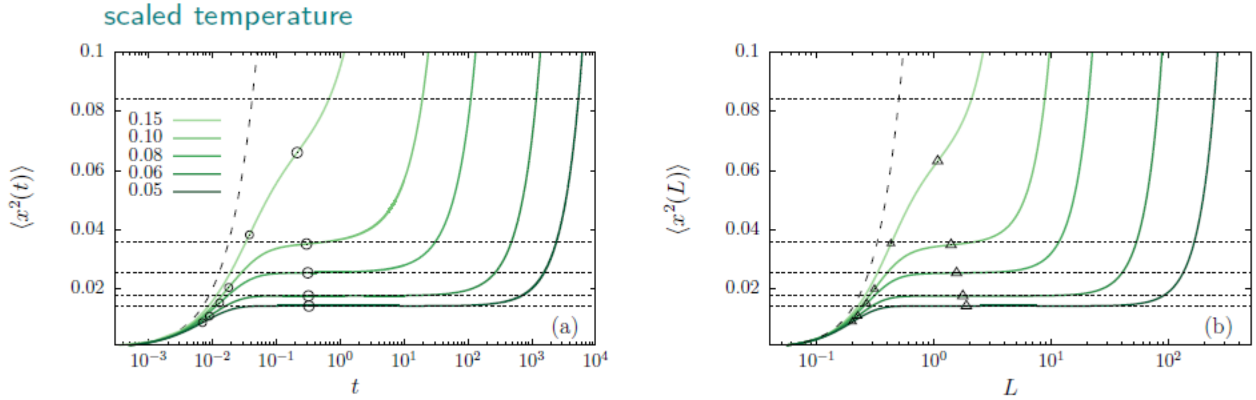
scaled temperature



$$\langle x^2 \rangle_{NQE} = \langle x^2(t^*) \rangle = \langle x^2(L^*) \rangle$$



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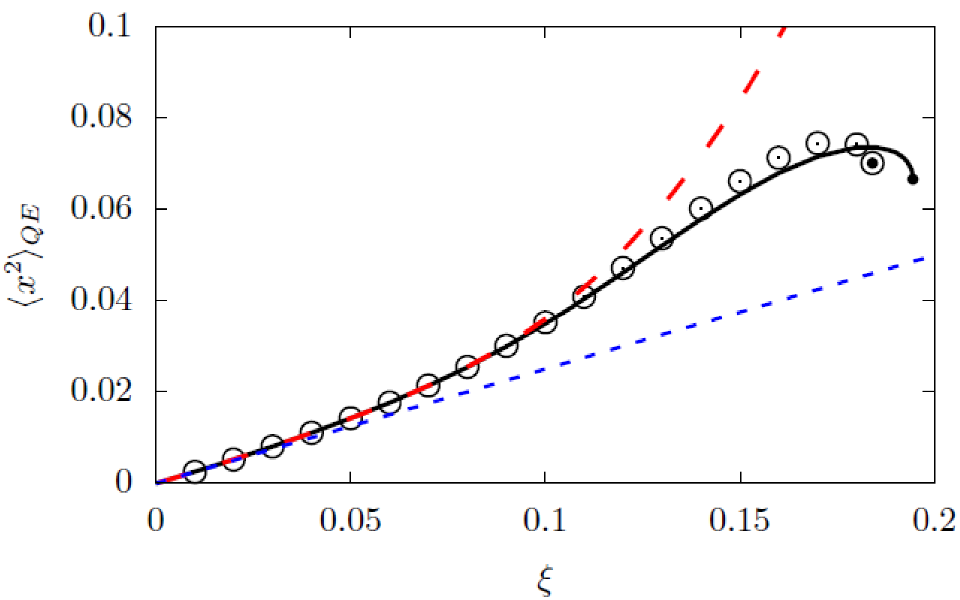
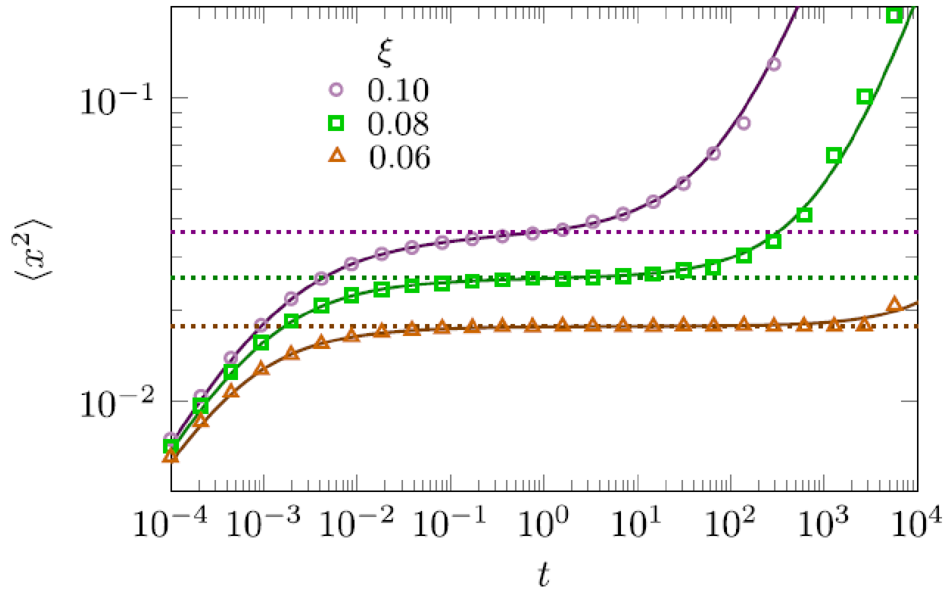


A comparison between the log-inflection in *L* (black solid line) and in time (circles). Harmonic (blue) and small xi (red) approxs.

Variant with anomalous diffusion

$$\frac{\partial}{\partial t}P \rightarrow {}^{C}D_t^{\alpha}P \equiv \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-t')^{\alpha}}\frac{dP}{dt'}dt'. \qquad \text{for } 0 < \alpha < 1$$

$$\langle (x - \langle x \rangle)^2 \rangle \sim t^{\alpha}$$



CTRW (continuous time random walk)

- jumps $p_j = \frac{1}{2}(1 \pm g_j)$,
- waiting times $\psi(t) = \alpha\tau^{-1}(\tau/t)^{\alpha+1}$, for $t > \tau$

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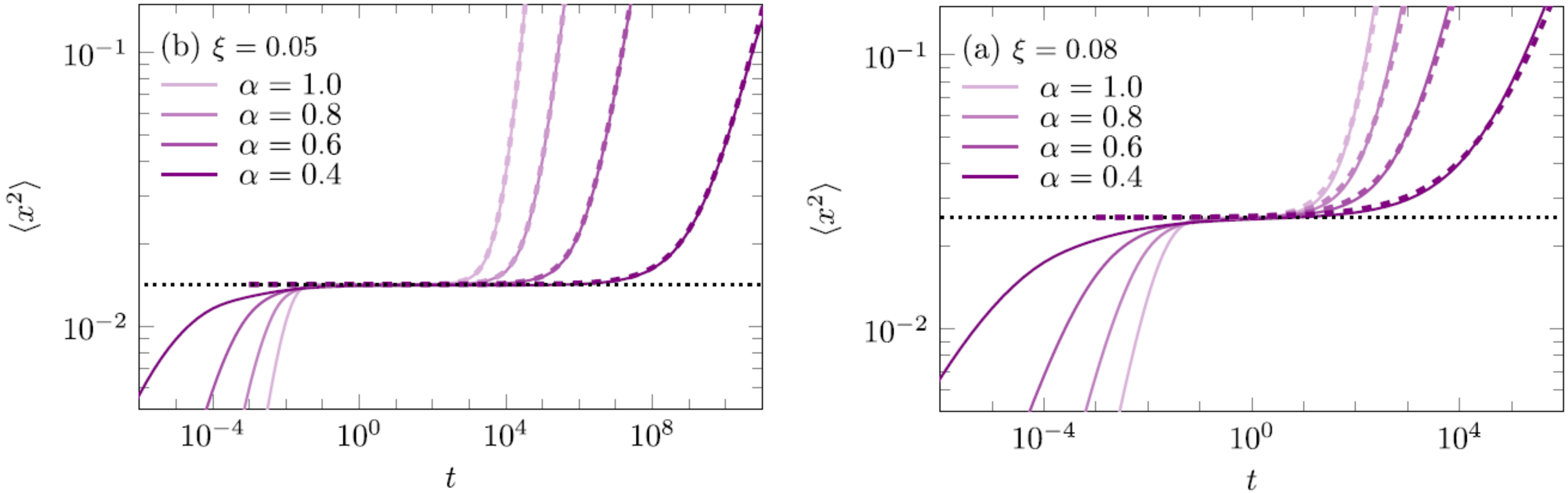
Variant with anomalous diffusion

$$\frac{\partial}{\partial t}P \to {}^{C}D_t^{\alpha}P \equiv \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-t')^{\alpha}}\frac{dP}{dt'}dt', \qquad \text{for } 0 < \alpha < 1$$

- Subdiffusion extends the duration of NNQE



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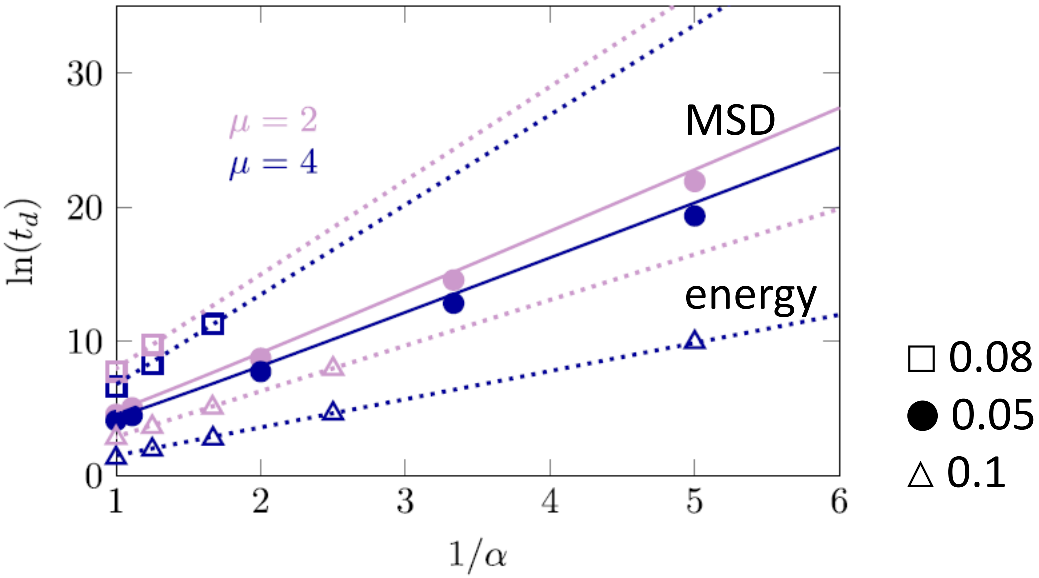
Variant with anomalous diffusion

$$\frac{\partial}{\partial t}P \to {}^{C}D_t^{\alpha}P \equiv \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-t')^{\alpha}}\frac{dP}{dt'}dt'. \qquad \text{for } 0 < \alpha < 1$$

- Subdiffusion extends the duration of NNQE $t_d = t^{**}(\delta/100)^{\frac{2}{3\alpha}} \sim (e^{\frac{1}{\xi}}\delta/100)^{\frac{2}{3\alpha}}$



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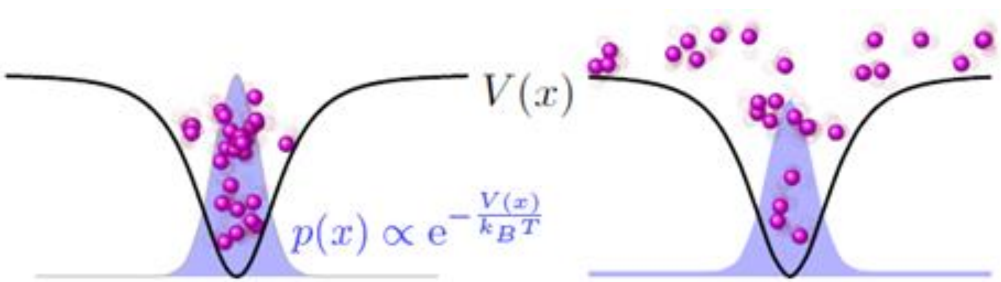
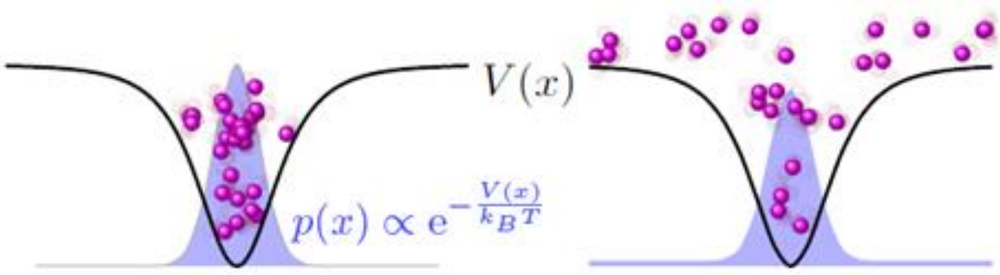
Final remarks

- We have shown that for asymptotically flat potentials, it is still possible to apply the tools from BG-statistical mechanics, using proper regularization.
- The proposed regularization process is simple and capable of accessing the NQE values accurately.
- Rigorous results were obtained through an eigen-function expansion (corrections to almost time-independent solution).

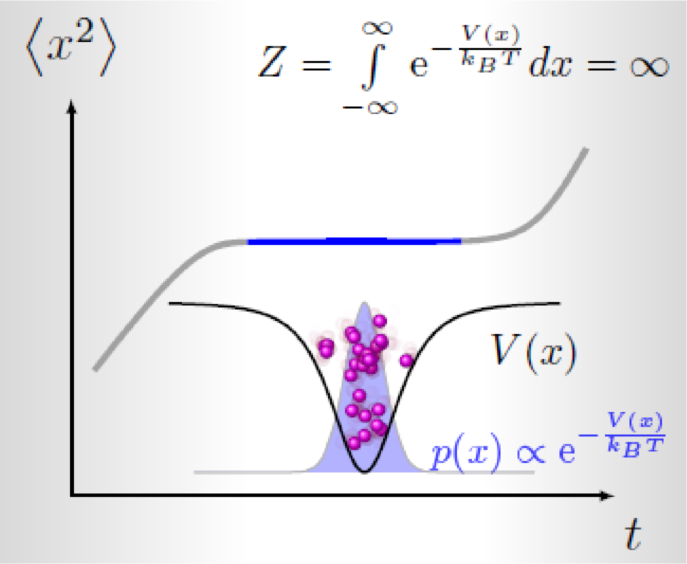


Perspectives

- Relaxation properties, fluctuation-dissipation relations Green–Kubo relations, etc.
- Variant with stochastic resetting.
- We have extended the formalism to fractional dynamics (time derivative), but other variants remain to be done.
- Interacting particles problem.



Thank you!



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Entropy 23, 131 (2021)
Phys Rev Res 2, 043088 (2020)

Κωνσταντίνο,

Χαρούμενα γενέθλια!

Συγχαρητήρια!

και σας ευχαριστώ!