On the Connection Between the q-Exponential Probability Distributions and the S_q Entropy

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Some of the main *themes* along which research on S_q-thermostatistics was to be conducted where already identified during the period 1993-1997.

- Long-range interactions
- Systems involving power-law density-dependent diffusion
- Edge of chaos; Weak chaos
- Micro-canonical description of finite systems

The S_q-Microcanonical connection has interesting historical, pedagogical, and conceptual facets

The q-distributions were already present at the very beginning of Statistical Mechanics.

The q-distributions are not exotic. They arise inevitably from basic mechanics

The S_q-microcaninical connections suggests an alternative route, still largely unexplored, to explain the phenomenological success of S_q-thermostatistics

Micro-Canonical Path Towards the q-Distribution

Composite system with weakly coupled subsystems A and B.

Gibbs micro-canonical ensemble with total energy E_T.

Level distribution of B quasi-continuous, with the number of states with energies less or equal to E growing as E^η.

The marginal probabilities of A are q-canonical, with $q = \frac{\eta}{\eta - 1}$ and $\beta = \frac{\eta - 1}{E_T}$.

A. Plastino and A.R. Plastino, *PLA* **193** (1994) 140.

When q-Exponentials Lacked a Name

"*the world was so recent that many things lacked names, and in order to indicate them it was necessary to point*", Gabriel Garcia Marques, *One Hundred Years of Solitude*.

J. C. Maxwell, "*On Boltzmann's Theorem on the Average Distribution of Energy in a System of Material Points*" (1879).

Marginal probability density in configuration space,

$$\mathcal{F}(b_1, \dots, b_n) = \mathcal{C}\left(1 - \frac{V(b_1, \dots, b_n)}{E}\right)^{\frac{n-2}{2}}, \; q = \frac{n}{n-2}.$$

A popular derivation of the canonical distribution goes through the q-canonical one: Feynman, R. *Statistical mechanics: A set of lectures* (1972).

Following in Maxwell's footsteps, more recent authors have, unawarely, depicted nice q-distributions:

J. R. Ray and H. W. Graben, *Small systems have non-Maxwellian momentum distributions in the microcanonical ensemble*, *Phys. Rev. A* **44** (1991) 6905.

F. L. Román, J. A. White, and S. Velasco, *Microcanonical single-particle distributions for an ideal gas in a gravitational field*, *Eur. J. Phys.* **16** (1995) 83.

Why Tsallis statistics? – Baranger's Conjecture

Baranger (2002) suggested that the microcanonical scenario advanced by Plastino & Plastino (1994) may be extended to non-equilibrium processes.

A subsystem may interact with a finite number of effective degrees of freedom of the rest of the system, establishing a q-canonical quasi-equilibrium state.

Possible connection with violent relaxation in galactic dynamics (Lynden-Bell, 1967).

M. Baranger, Why Tsallis statistics?. *Physica A* **305** (2002) 27.

The AMAA formulation of the S_q-micro-canonical connection for finite classical Hamiltonian systems with a continuous phase-space

$$H = H_A(\omega_A) + \sum_{k=1}^{J} H_k(\omega_k),$$

$$q = 1 + \left[\left(\sum_{k=1}^{J} \frac{n_k}{l_k} \right) - 1 \right]^{-1} .$$

The theoretically derived q-canonical distribution was neatly verified through numerical experiments on a chain of anharmonic oscillators.

A.B. Adib, A.A. Moreira, J.S. Andrade Jr, and M.F. Almeida, *Physica A* **322** (2003) 276.

Researchers have explored and elaborated the S_q-micro-canonical connection in a number of interesting directions.

V. Aquilanti, E.P. Borges, N.D. Coutinho, K.C. Mundim, and V.H. Carvalho-Silva, *From statistical thermodynamics to molecular kinetics: the change, the chance and the choice*, Rendiconti Lincei **29** (2018) 787.

J. A. S. Lima and A. Deppman, *Tsallis meets Boltzmann: q-index for a finite ideal gas and its thermodynamic limit*, *Phys. Rev. E* **101** (2020) 040102.

A.R. Plastino and A. Plastino, *Brief Review on the Connection between the Micro-Canonical Ensemble and the S_q-Canonical Probability Distribution*, Entropy **25** (2023) 591.

Alternatives to the S_q-entropy representation of the q-distributions

J. Naudts and M. Baeten, *Non-extensivity of the configurational density distribution in the classical microcanonical ensemble*, *Entropy* **11** (2009) 285.

J. D. Ramshaw, *Maximum entropy and constraints in composite systems*, *PRE* **105** (2022) 024138.

Are generalized entropies needed, in order to get appropriate optimum-entropy representations of probability distributions?

$$\text{Entropy} \longrightarrow S = -\int F(x) \ln F(x)\, dx.$$

$$\text{Constraints} \longrightarrow b = \langle B \rangle = \int B(x) F(x)\, dx; \int F(x)\, dx = 1.$$

$$\text{Entropy optimization} \longrightarrow F(x) = \exp(-1 - \alpha - \beta B(x)).$$

$$\text{Logarithmic Constraints} : F(x) \longrightarrow B(x) = \delta - \ln F.$$

$$\text{Entropy optimization} \longrightarrow F(x;\beta) = \frac{[F(x)]^{\beta}}{\int [F(x)]^{\beta} dx}.$$

For pure power laws, the Lagrange multiplier corresponding to the logarithmic constraint coincides with the power-law exponent

$$\text{Logarithmic Constraints}: \longrightarrow \langle \ln n \rangle.$$

$$\text{Entropy optimization} \longrightarrow P(n) = A n^{-\beta}.$$

Possible justification of logarithmic constraints: logarithmic response of perception; Weber-Fechner law (examples: astronomical magnitudes; decibels).

M. Visser, *Zipf's law, power laws and maximum entropy*, New Jour. Phy. **15** (2013) 043021.

The entropy optimization game with one constraint (besides normalization)

Choose a trace-form entropy $S_G[F] = \int \mathcal{C}[F(x)]dx$, a relevant constraint $\langle B(x) \rangle$, and the appropriate phase space variables x.

Given the sole input $b = \langle B(x) \rangle$, determine the probability distribution optimizing the entropy S_G:

$$F(x) \; = \; \mathcal{A}[-\alpha - \beta x]$$

The function $\mathcal{A}$ is determined by the function $\mathcal{C}$.

The Inverse Game: finding economical entropic characterizations

Given a parameterized family of probability distributions

$$F(x) = \mathcal{A}[-\alpha - \beta x]$$

find a trace-form entropy S_G and an appropriate constraint $\langle B(x) \rangle$ yielding a representation in terms of entropy optimization.

The inverse game has a unique answer. The forms of S_G and $B(x)$ are completely determined by the form of the parameterized family of probability distributions.

A.R. Plastino, A. Plastino, and B.H. Soffer, *Ambiguities in the forms of the entropic functional and constraints in the*

Pure power laws and their relatives

$$P(n) = A\, n^{-\beta} \longrightarrow \left\{ \begin{array}{c} S[P] = -\sum P(n) \ln P(n) \\ \langle \ln n \rangle \end{array} \right.$$

$$P(n) = A\, a_n^{-\beta} \longrightarrow \left\{ \begin{array}{c} S[P] = -\sum P(n) \ln P(n) \\ \langle \ln a_n \rangle \end{array} \right.$$

From the point of view of the entropy optimization game, pure power laws and Gibbs distributions are close relatives.

The q-exponential family

The q-exponential family of probability distributions or densities is *bi-parametric*: it depends on *two* parameters, β and q.

Depending on which parameter we regard as variable, and which as fixed, the entropy optimization game will determine the appropriate entropy.

- q fixed and β variable $\longrightarrow S_q$
- β fixed and q variable $\longrightarrow S_1$

The second case, however, requires a β-dependent constraint of the form $\langle 1-(1-q)\beta E \rangle$.

Final Remarks

- q-distributions are not exotic
- q-distributions are inextricably linked to some basic, early ideas of statistical mechanics
- The micro-canonical connection is the less unexplored road to q-statistics. Links between this connection, and other roads leading to q-statistics are worth exploring
- Once we accept the entropy optimization game, the presence of q-distributions inevitably leads to S_q entropies

HAPPY BIRTHDAY CONSTANTINO !!!!