STATISTICAL MECHANICS FOR COMPLEXITY

A CELEBRATION OF THE 80TH BIRTHDAY OF CONSTANTINO TSALLIS

RIO DE JANEIRO, 6 TO 10 NOVEMBER 2023

Footprints of q-Statistics at the 'Edge of Chaos'

Other fragments from the last 20-years of collaboration with Constantino

Alessandro Pluchino

Department of Physics and Astronomy "E.Majorana"
University of Catania and INFN sezione di Catania, Italy

2002

work-time-line

International Workshop

Anomalous distributions, non-linear dynamics and non-extensivity

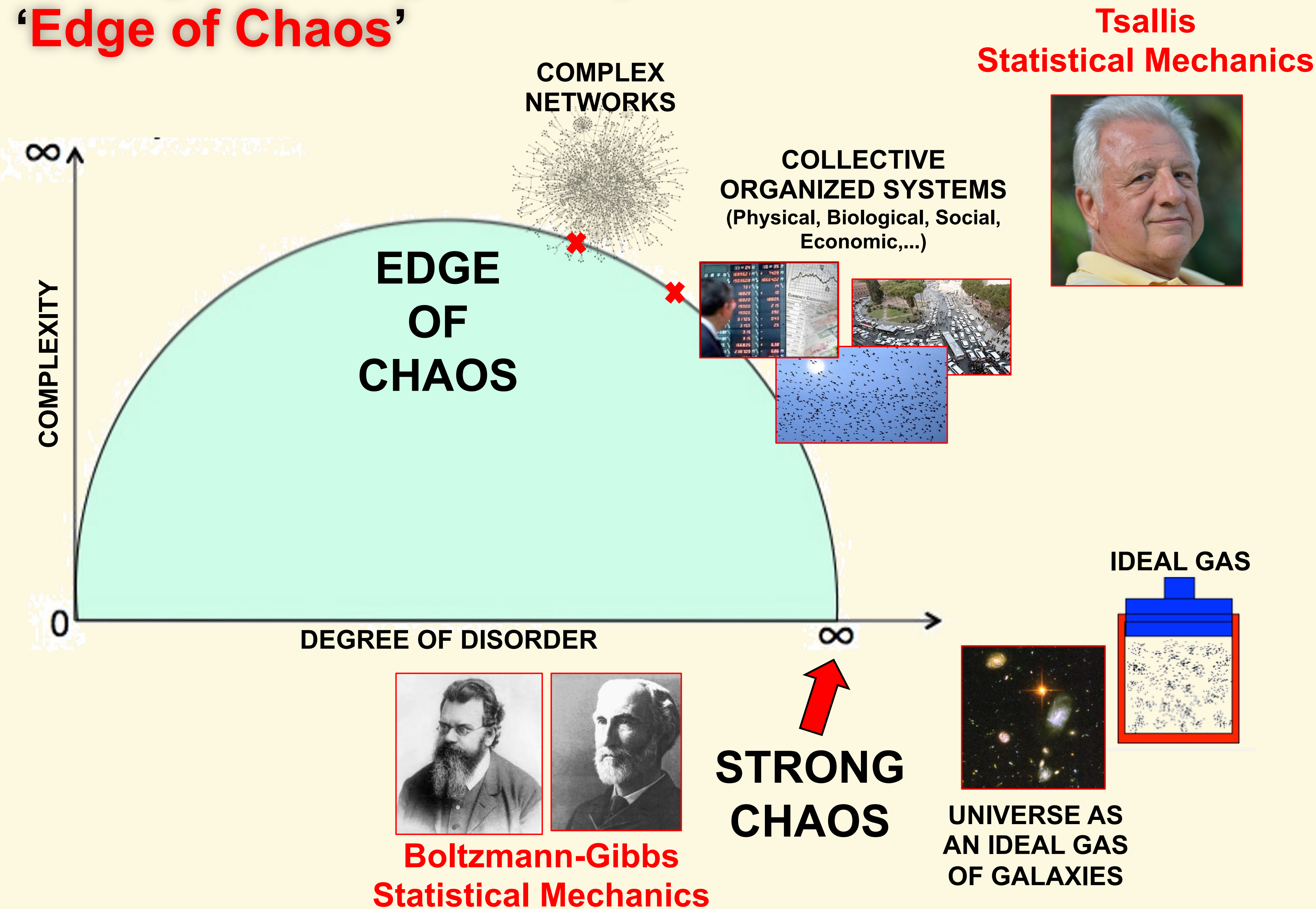
Santa Fè, New Mexico (USA)

Econophysics Colloquium

Lipari (Italy)

2023

A Long-standing Journey at the 'Edge of Chaos'



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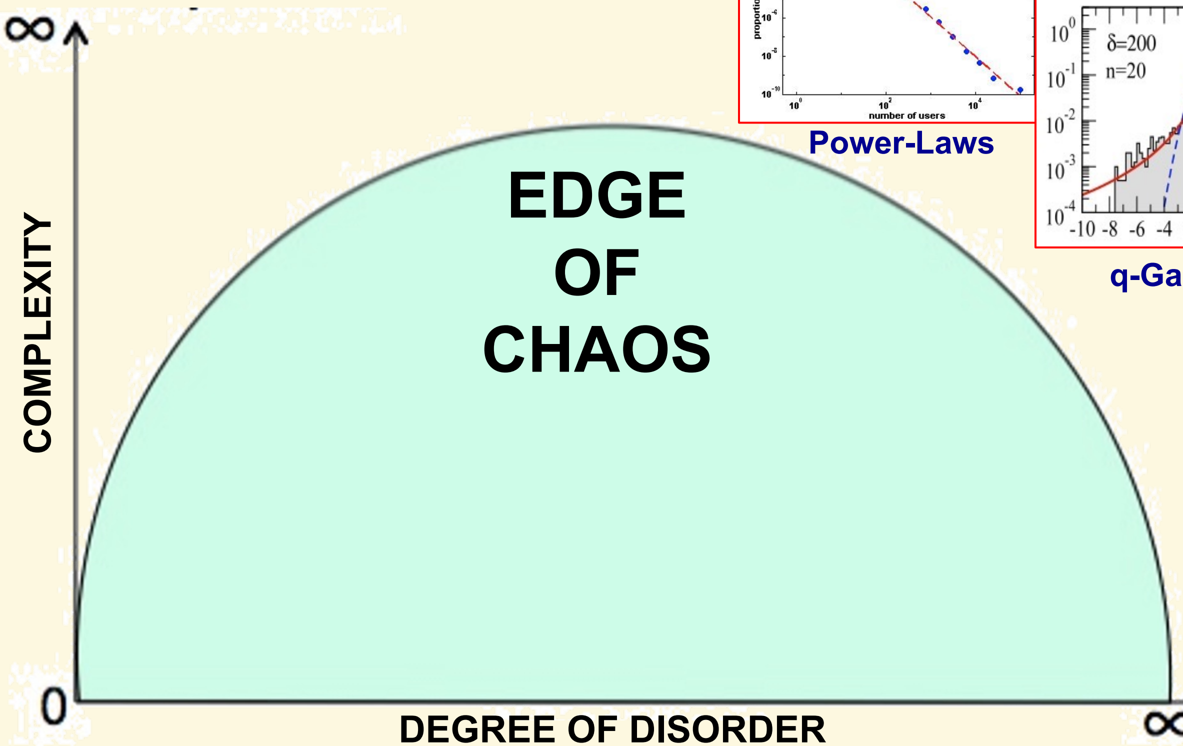
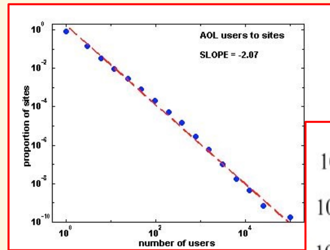
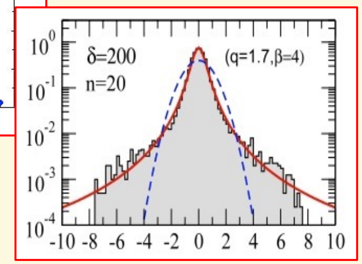


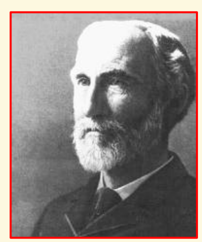


Power-Laws



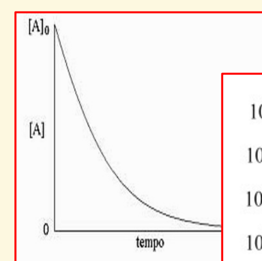
q-Gaussians

Tsallis Statistical Mechanics

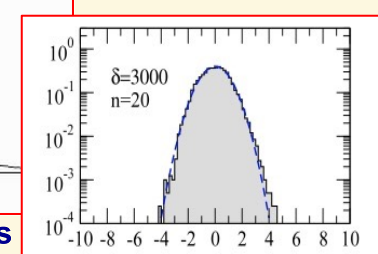


Boltzmann-Gibbs Statistical Mechanics

STRONG CHAOS



Exponentials



Gaussians

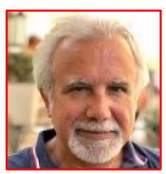
A Long-standing Journey at the ‘Edge of Chaos’

Tsallis Statistical Mechanics

UNIVERSITÀ DEGLI STUDI DI CATANIA
DOTTORATO DI RICERCA IN FISICA

ALESSANDRO PLUCHINO

METASTABILITY, NONEXTENSIVITY
AND GLASSY DYNAMICS
IN A CLASS OF LONG-RANGE HAMILTONIAN
MODELS



TUTOR: CHIAR.MO PROF.A.RAPISARDA

TESI PER IL CONSEGUIMENTO DEL TITOLO

XVII CICLO, 2001-2004

**My PhD Thesis
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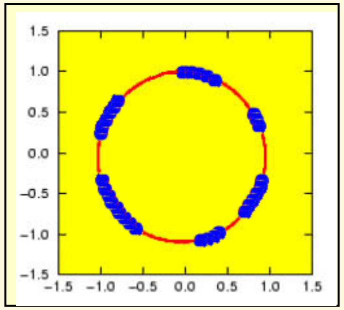

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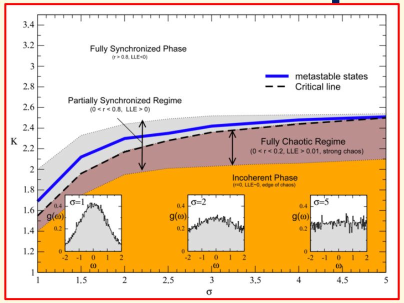
Second Step



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symmetry
IMPACT FACTOR 2.645
CITESCORE 2.5 SCOPUS
Special Issue
"Hamiltonian and Overdamped Complex Systems"
Guest Editors:
Prof.Dr. Antonio Rodriguez, Prof.Dr. Alessandro Pluchino, Prof.Dr. Ugur Tirnakli
Deadline for manuscript submissions: **31 March 2024**

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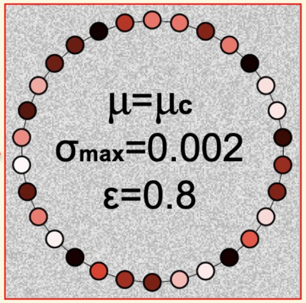

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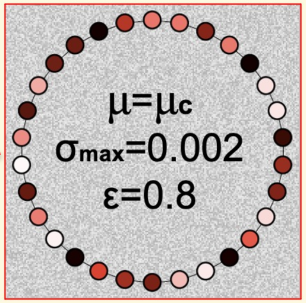

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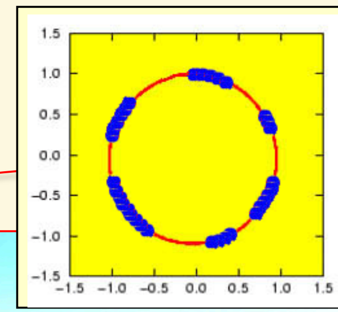
2002

work-time-line

2004-2006



2023

Hamiltonian Mean-Field (HMF) Model and the Statistical Mechanics of Coupled Oscillators

$$H = K + V = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - \cos(\vartheta_i - \vartheta_j)]$$

Antoni and Ruffo PRE 52 (1995) 2361

order parameter

$$\vec{M} = \frac{1}{N} \sum_{i=1}^{N} \vec{S}_i = M e^{i\phi}$$

$$F = free\ energy\ density$$

$$\beta = \frac{1}{T} \quad (k_B = 1)$$

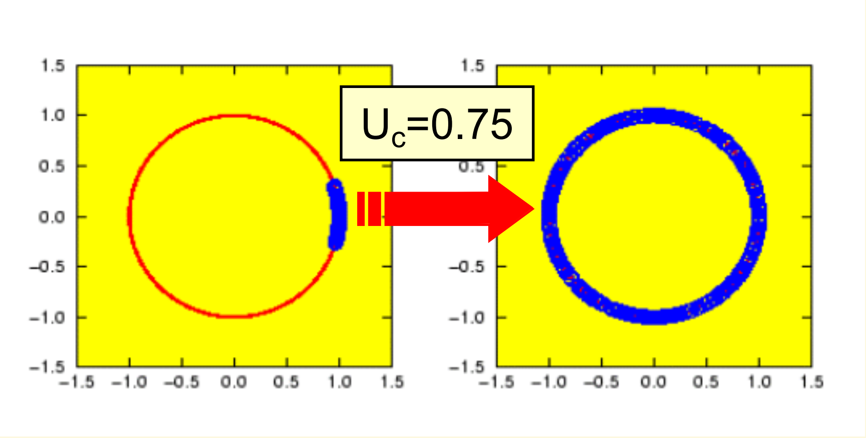
$$T = temperature$$

Caloric Curve

$$U = \frac{H}{N} = \frac{\partial(\beta F)}{\partial \beta} = \frac{1}{2\beta} + \frac{1}{2}(1 - M^2)$$

AT EQUILIBRIUM...

$M \sim 1$
$U < U_c$
Condensed Phase



$M \sim 0$
$U > U_c$
Homogeneous Phase

Hamiltonian Mean-Field (HMF) Model and the Statistical Mechanics of Coupled Oscillators

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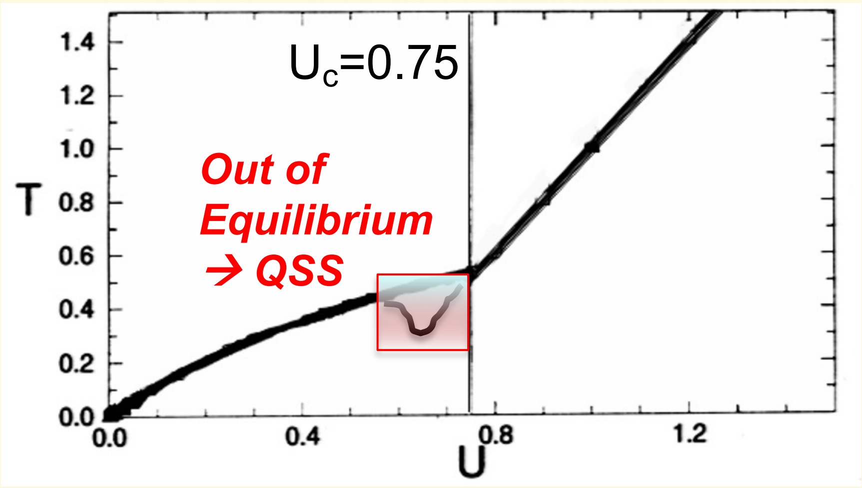
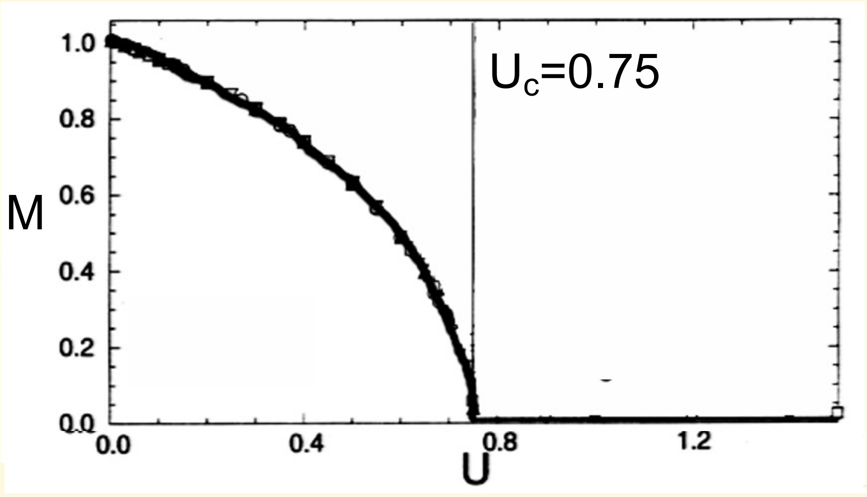
AT EQUILIBRIUM...

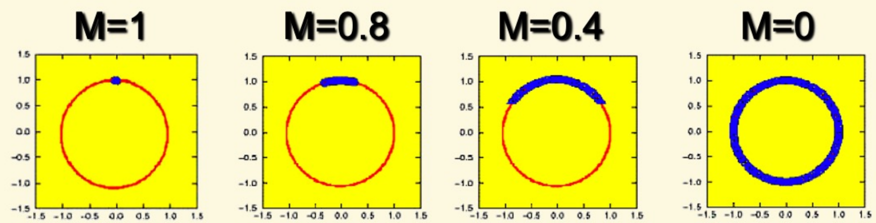




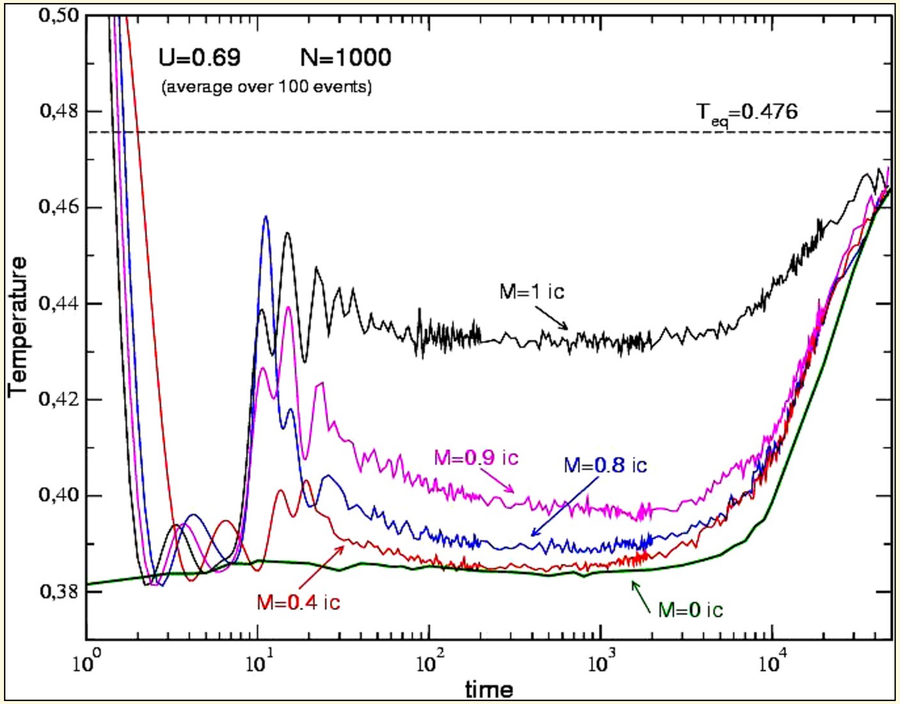
Correlations and Glassy dynamics in the QSS Regime of the HMF Model

Decoupled equations of motion:

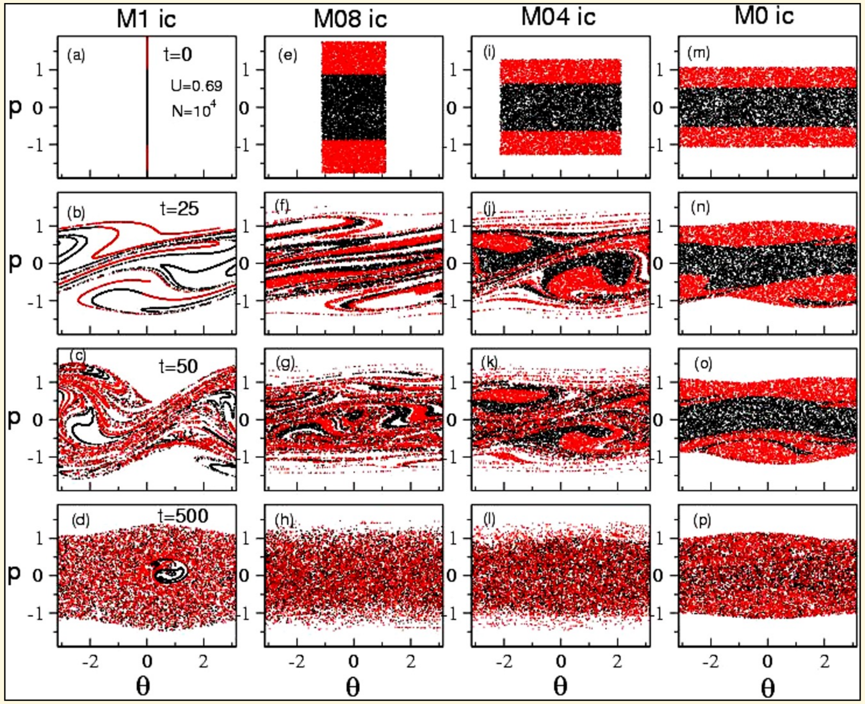
$$\ddot{\theta}_i + M\sin(\theta_i - \phi) = 0 \qquad i = 1, \ldots, N$$



QSS Regime for several IC



μ-Space Correlations for several IC



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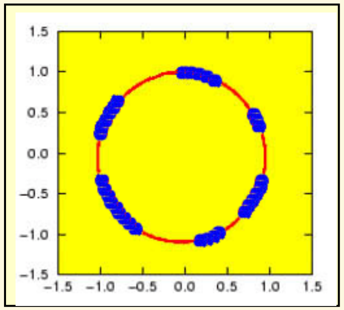
M1 IC

M0 IC

Clusters Competition

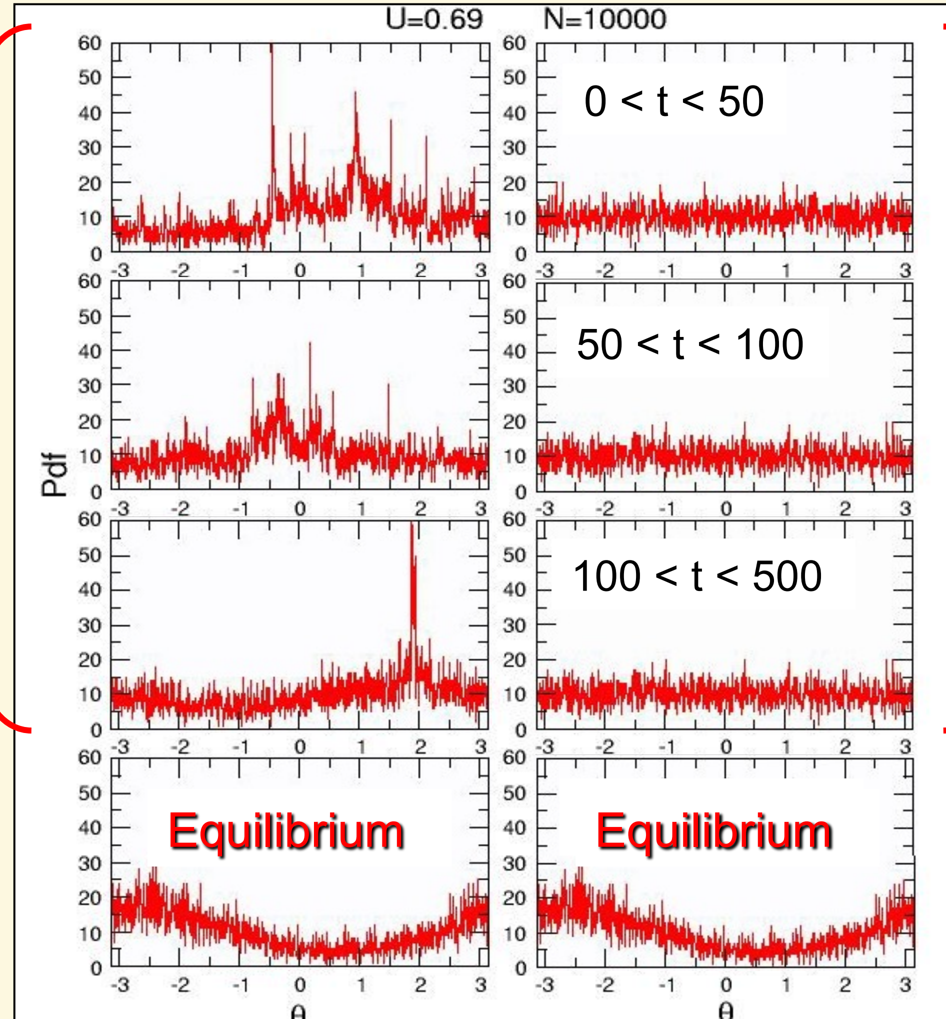


NON EXTENSIVE EFFECTS

Andrea's Talk

DYNAMICAL FRUSTRATION

QSS Regime



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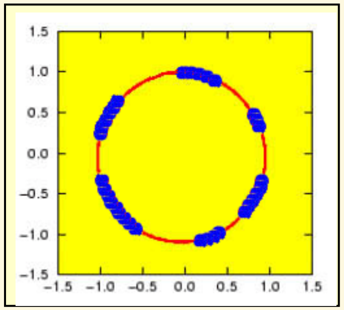
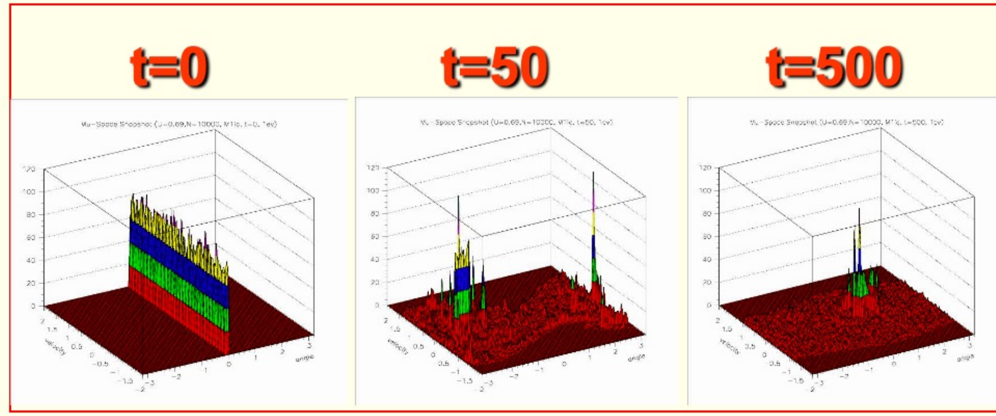


NON EXTENSIVE EFFECTS

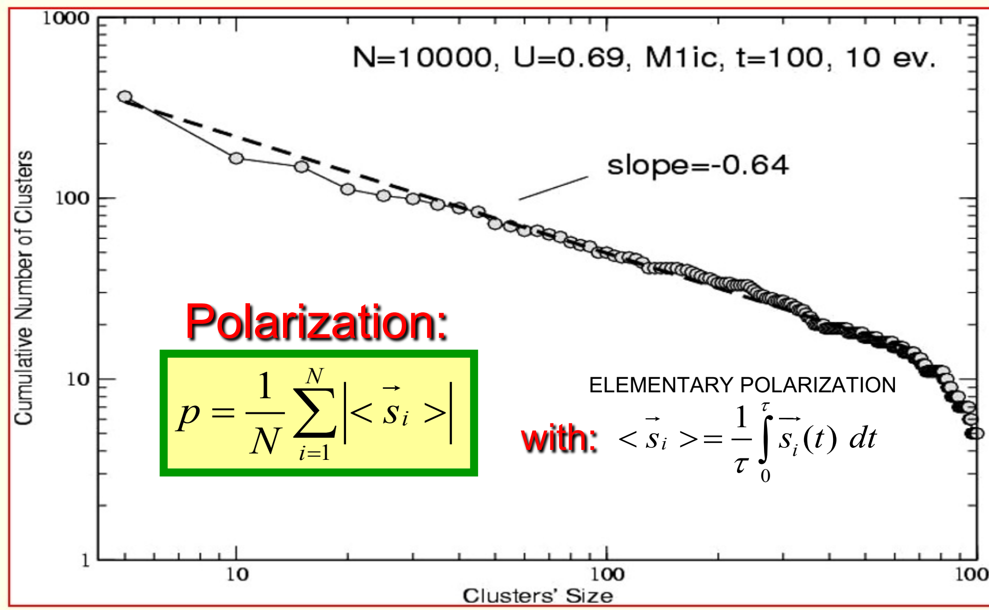
Andrea's Talk

DYNAMICAL FRUSTRATION

WEAK ERGODICITY BREAKING



Cumulative Number of Clusters



Polarization:

$$p = \frac{1}{N}\sum_{i=1}^{N}\left|<\vec{s}_i>\right|$$

ELEMENTARY POLARIZATION

with: $<\vec{s}_i> = \frac{1}{\tau}\int_0^{\tau}\vec{s}_i(t)\,dt$

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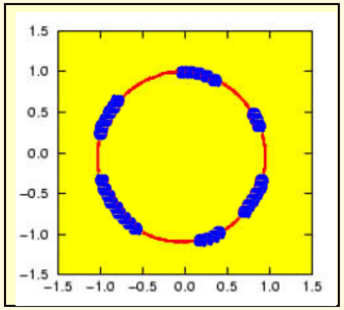


NON EXTENSIVE EFFECTS

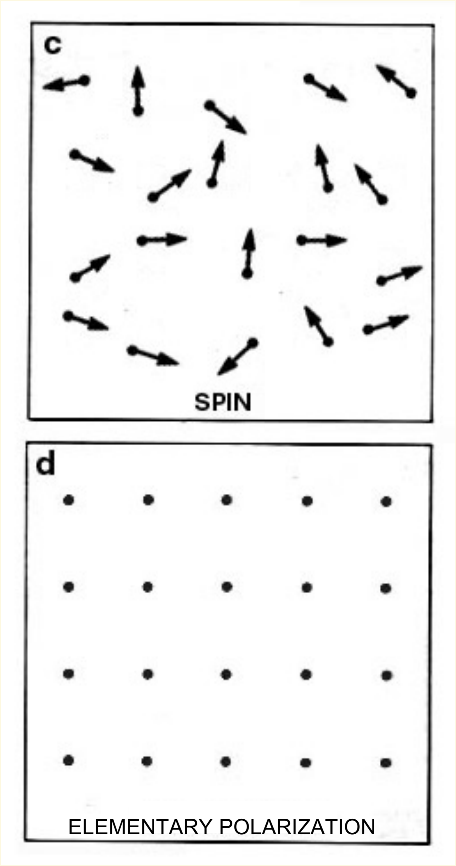
Andrea's Talk

DYNAMICAL FRUSTRATION

WEAK ERGODICITY BREAKING

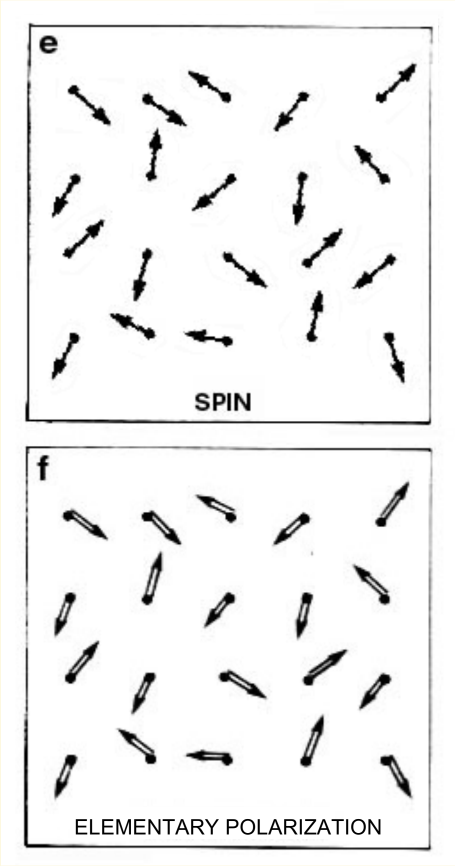
HOMOGENEOUS PHASE:

$M = p \sim 0$



SPIN GLASS PHASE:

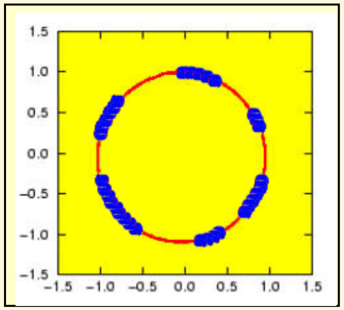
$M \sim 0 \qquad p \neq 0$



A Pluchino, A Rapisarda, Glassy Dynamics and Nonextensive effects in the HMF model: the importance of initial conditions.
Progress of Theoretical Physics vol. 162, pp. 18-28 (2006)

Correlations and Glassy dynamics in the QSS Regime of the HMF Model

Clusters Competition



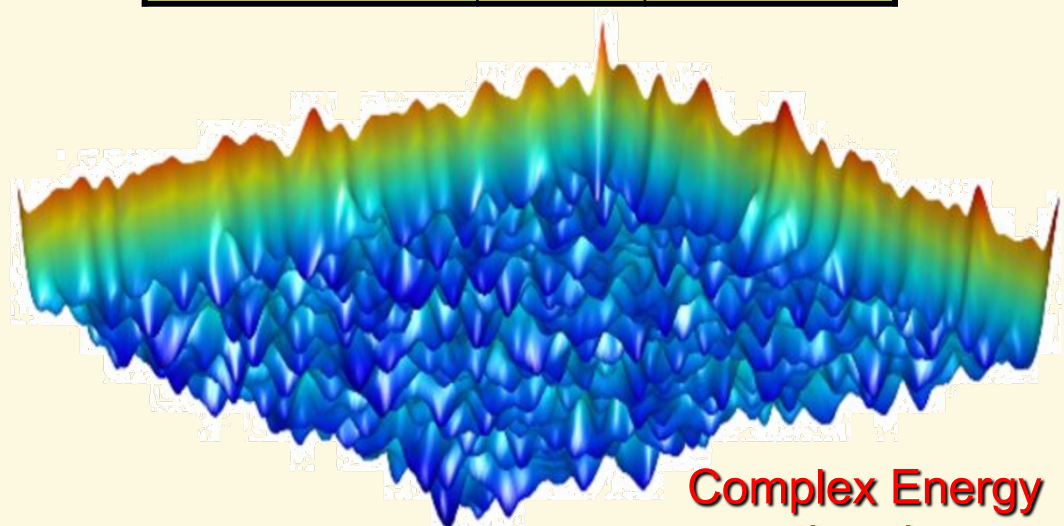
Phase	*Order Parameters*	
Condensed	$p = M \neq 0$	
Homogeneous	$p = M \sim 0$	
SPIN GLASS (QSS regime)	$p \neq 0$	$M \longrightarrow 0$

NON EXTENSIVE EFFECTS

Andrea's Talk

DYNAMICAL FRUSTRATION

WEAK ERGODICITY BREAKING



Complex Energy Landscape

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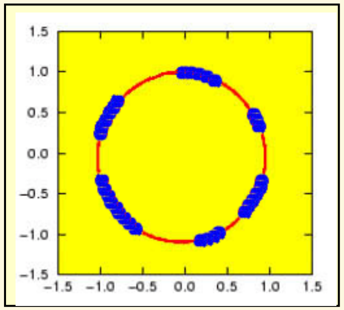
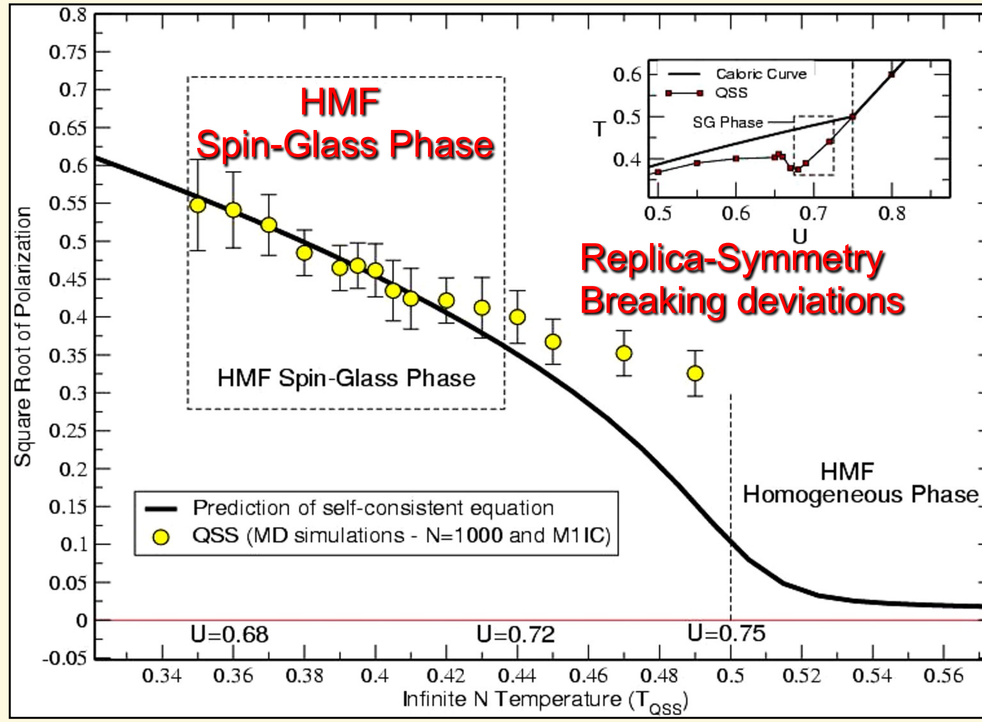


NON EXTENSIVE EFFECTS

Andrea's Talk

DYNAMICAL FRUSTRATION

WEAK ERGODICITY BREAKING → **REPLICA-SYMMETRIC SOLUTION FOR THE HMF SPIN-GLASS PHASE**



self-consistent equation

$$p = 1 - \sqrt{\frac{2}{p}}\,\beta^{-1}\int_0^\infty r^2\,dr\,\exp\left(-\frac{r^2}{2}\right)\frac{I_1\left[\beta r\sqrt{p/2}\right]}{I_0\left[\beta r\sqrt{p/2}\right]} \qquad (8)$$

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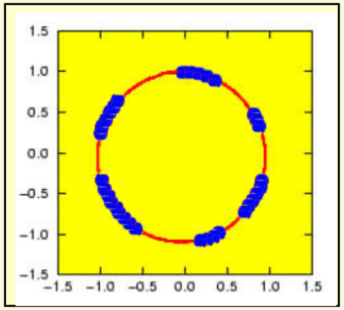

PRL **102,** 097202 (2009) PHYSICAL REVIEW LETTERS week ending 6 MARCH 2009

Generalized Spin-Glass Relaxation

R. M. Pickup,[1] R. Cywinski,[2,*] C. Pappas,[3] B. Farago,[4] and P. Fouquet[4]

[1]*School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, United Kingdom*
[2]*School of Applied Sciences, University of Huddersfield, Huddersfield HD1 3DH, United Kingdom*
[3]*Helmholtz Center Berlin for Materials and Energy, Glienickerstrasse 100, 14109, Berlin, Germany*
[4]*Institut Laue Langevin, 6 rue Jules Horowitz, 38000 Grenoble, France*

(Received 18 July 2008; published 4 March 2009)

Spin relaxation close to the glass temperature of CuMn and AuFe spin glasses is shown, by neutron spin echo, to follow a generalized exponential function which explicitly introduces hierarchically constrained dynamics and macroscopic interactions. The interaction parameter is directly related to the normalized Tsallis nonextensive entropy parameter q and exhibits universal scaling with reduced temperature. At the glass temperature $q = 5/3$ corresponding, within Tsallis' q statistics, to a mathematically defined critical value for the onset of strong disorder and nonlinear dynamics.

DOI: 10.1103/PhysRevLett.102.097202 PACS numbers: 75.50.Lk, 64.70.P−, 75.40.Gb, 76.60.Lz

NON EXTENSIVE EFFECTS

Andrea's Talk

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WEAK ERGODICITY BREAKING

REPLICA-SYMMETRIC SOLUTION FOR THE HMF SPIN-GLASS PHASE

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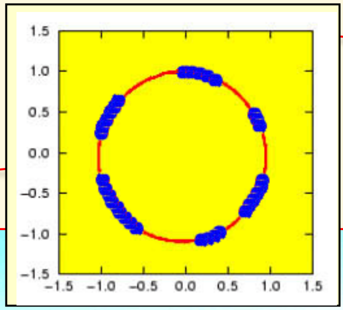
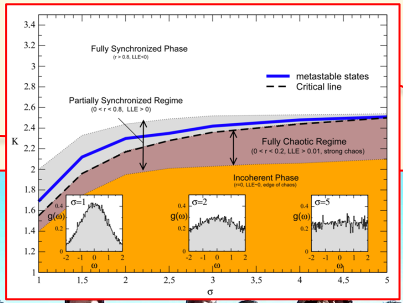
2002

work-time-line

2004-2006

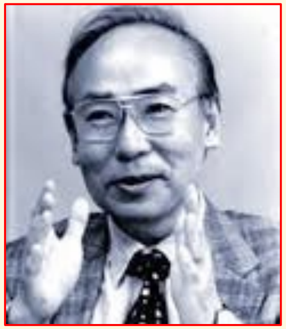


2009



2023

Kuramoto Model and the Synchronization of Coupled Oscillators

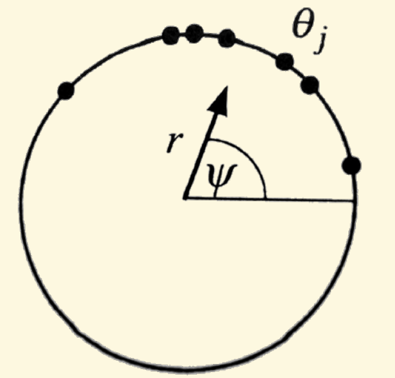


Yoshiki Kuramoto

coupling strenght

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

natural frequencies



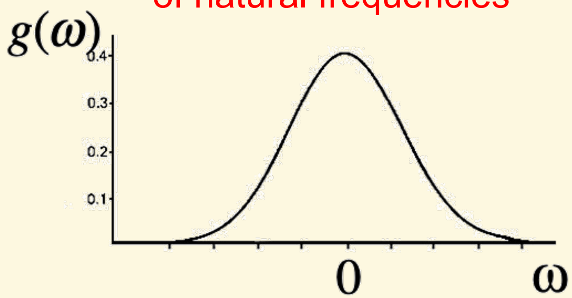
order parameter

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}.$$

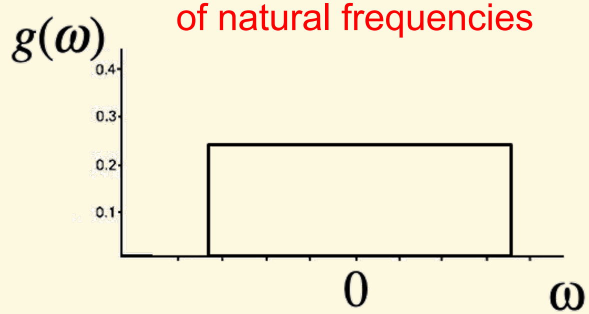
Decoupled equations:

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), \quad i = 1, \ldots, N$$

Gaussian Distribution
of natural frequencies

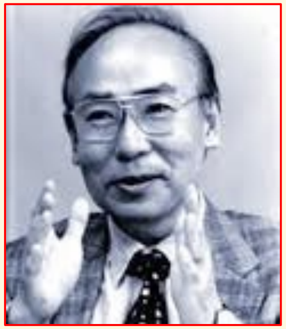


Uniform Distribution
of natural frequencies



Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, Springer, Berlin, 1984.

Kuramoto Model and the Synchronization of Coupled Oscillators



Yoshiki Kuramoto

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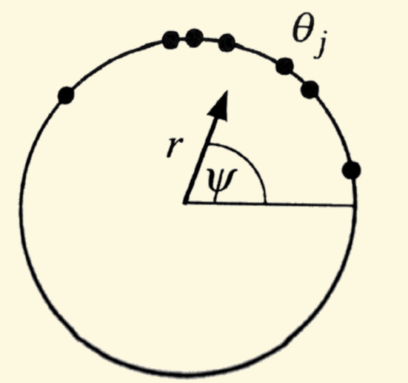
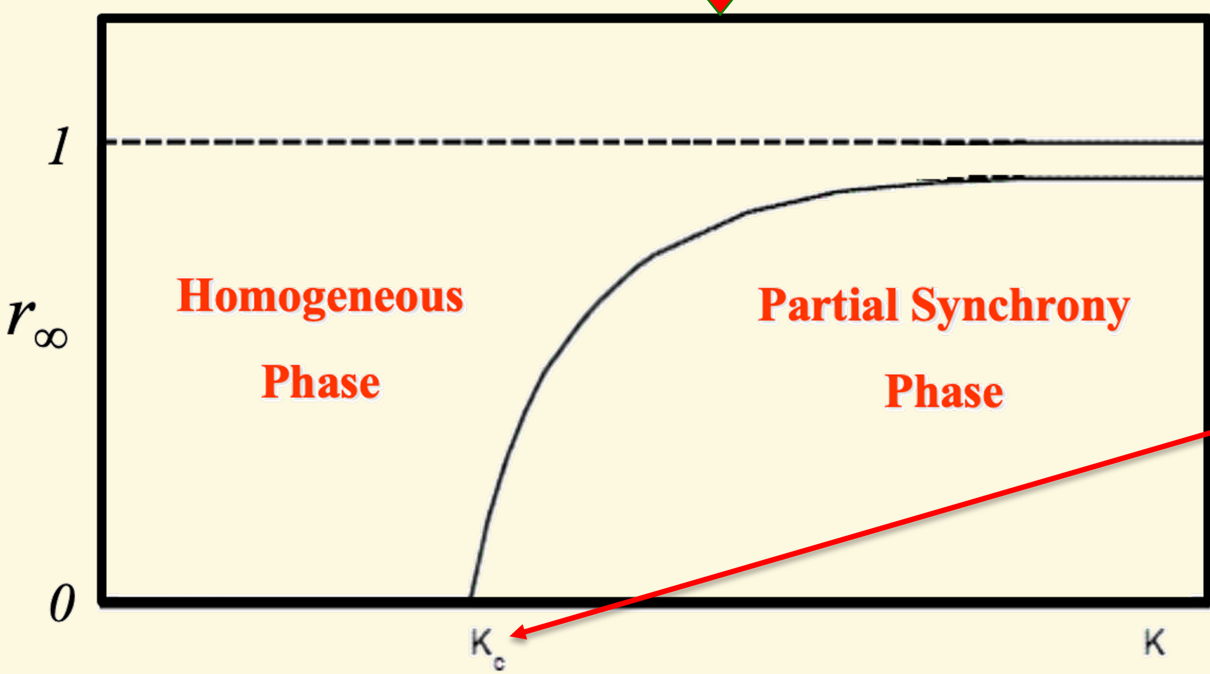


order parameter

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}.$$

critical coupling

$$K_C = \frac{2}{\pi g(0)}$$



Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, Springer, Berlin, 1984.

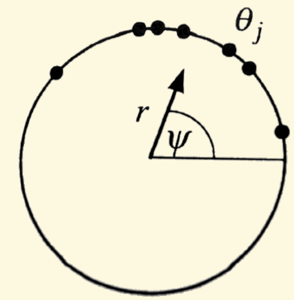
HMF and Kuramoto Model as limiting cases of a Damped-Driven model of Coupled Oscillators

coupling

Damped-Driven Oscillators

$$\ddot{\theta}_i + B\dot{\theta}_i + CM\sin(\theta_i - \phi) = \Gamma, \quad i = 1, \ldots, N,$$

damping — order parameter — driving



$$\mathbf{M} = Me^{i\phi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

HMF Model

$$\ddot{\theta}_i + M\sin(\theta_i - \phi) = 0$$

Conservative Limit:

$$B = 0$$

$$C = 1$$

$$\Gamma = 0$$

Kuramoto Model

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i)$$

Overdamped Limit:

$$B \gg 1$$

$$C = K$$

$$\Gamma = \omega$$

$$\mathrm{M} = r$$

G Miritello, A Pluchino, A Rapisarda, Phase Transitions and Chaos in Long-Range Models of Coupled Oscillators
Europhysics Letters vol. 85, pp. 10007 (2009)

Phase Transition and Chaos in the HMF Model

Damped-Driven Oscillators

coupling

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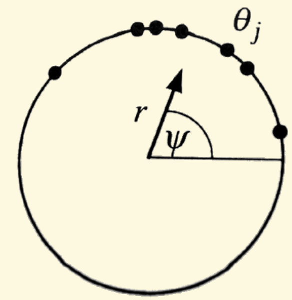
damping order parameter driving



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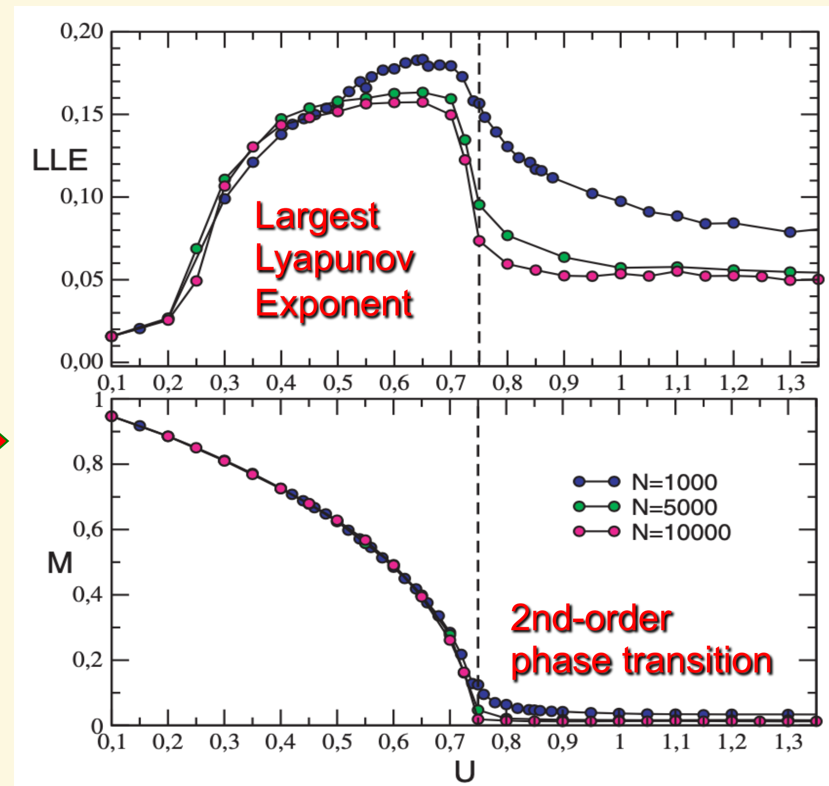

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Phase Transition and Chaos in the Kuramoto Model

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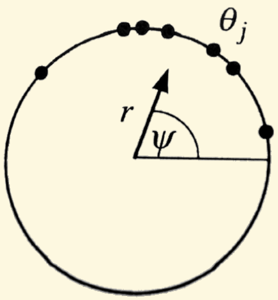

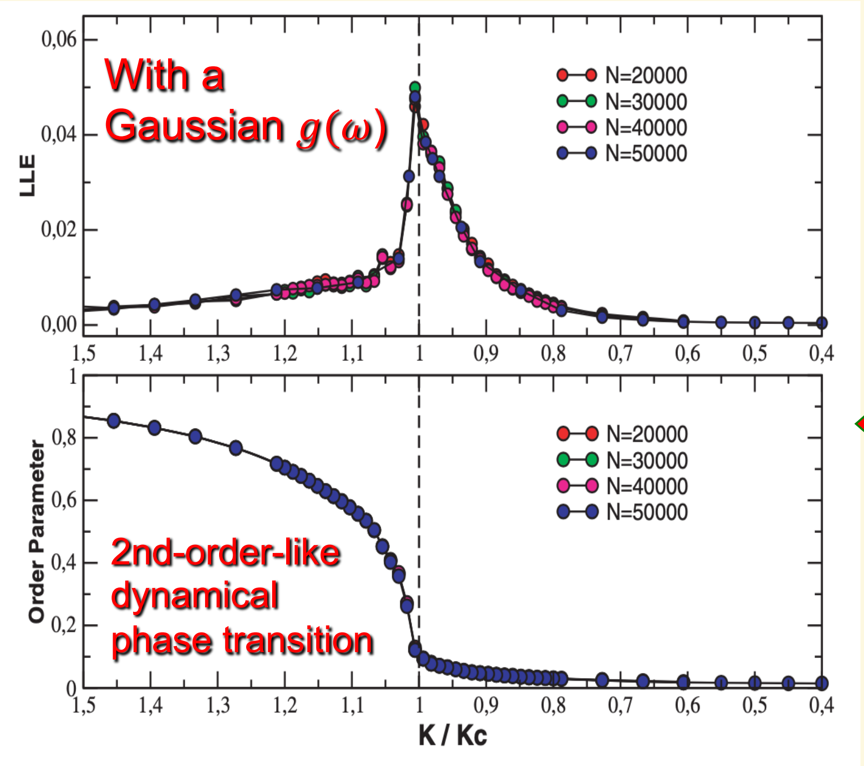

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Overdamped Limit:

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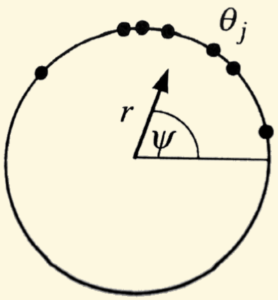
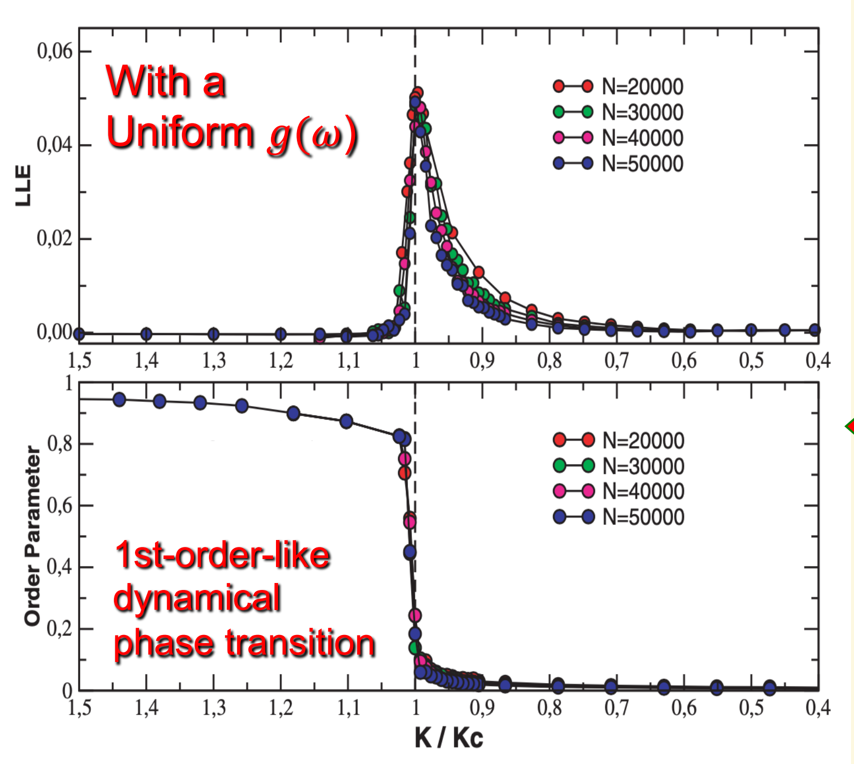
damping | order parameter | driving





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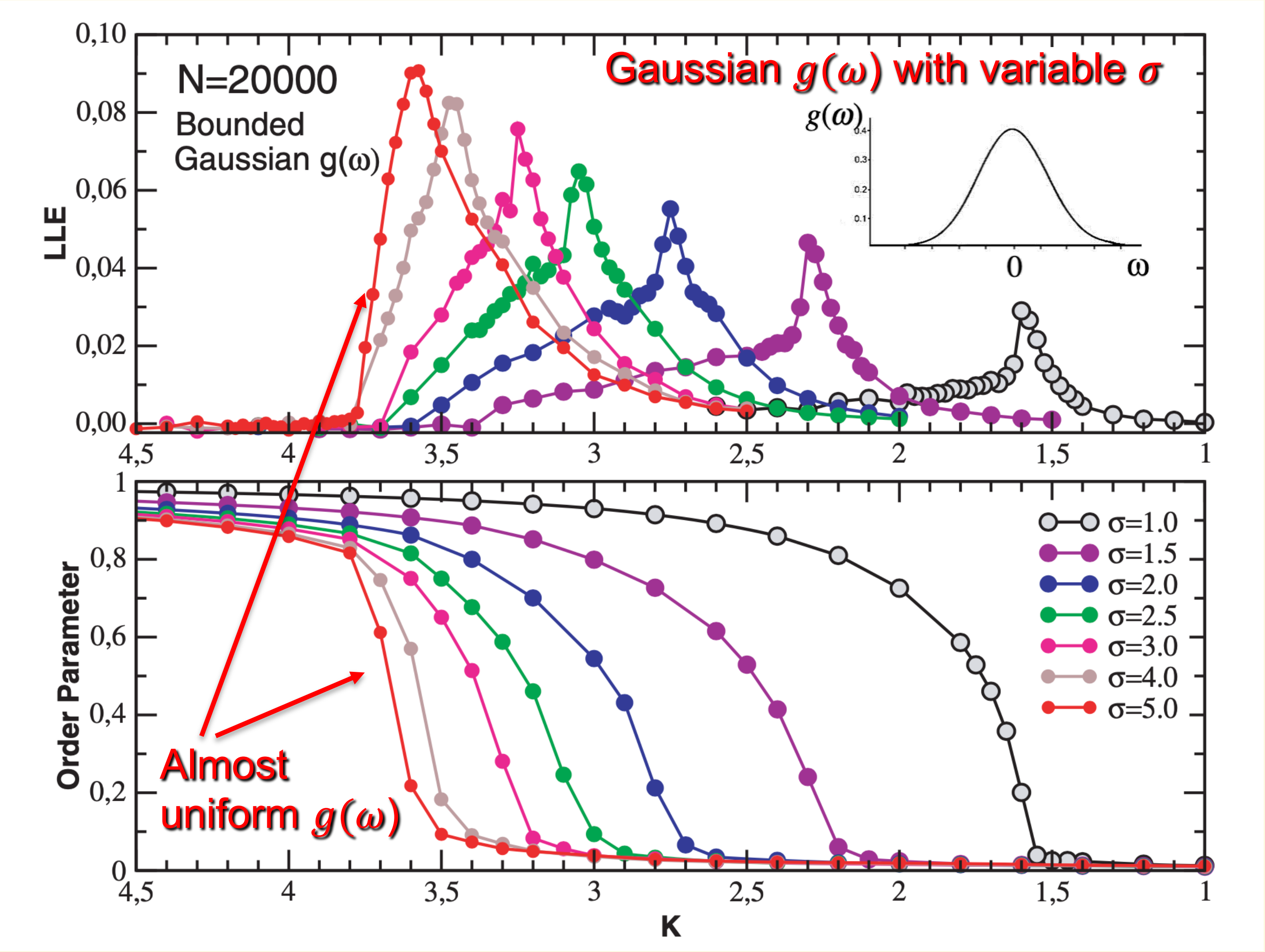

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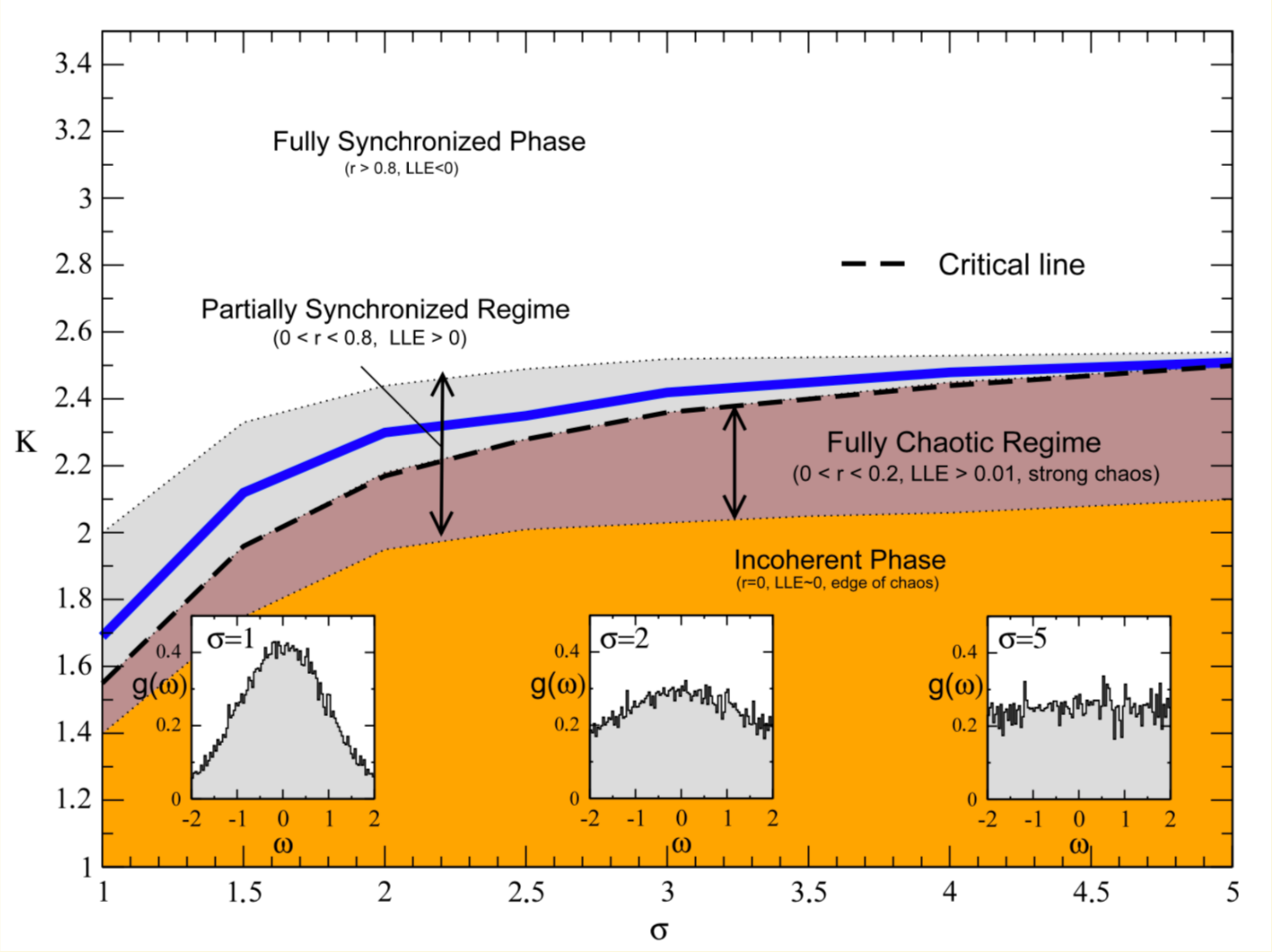
Phase Diagram of the Kuramoto Model



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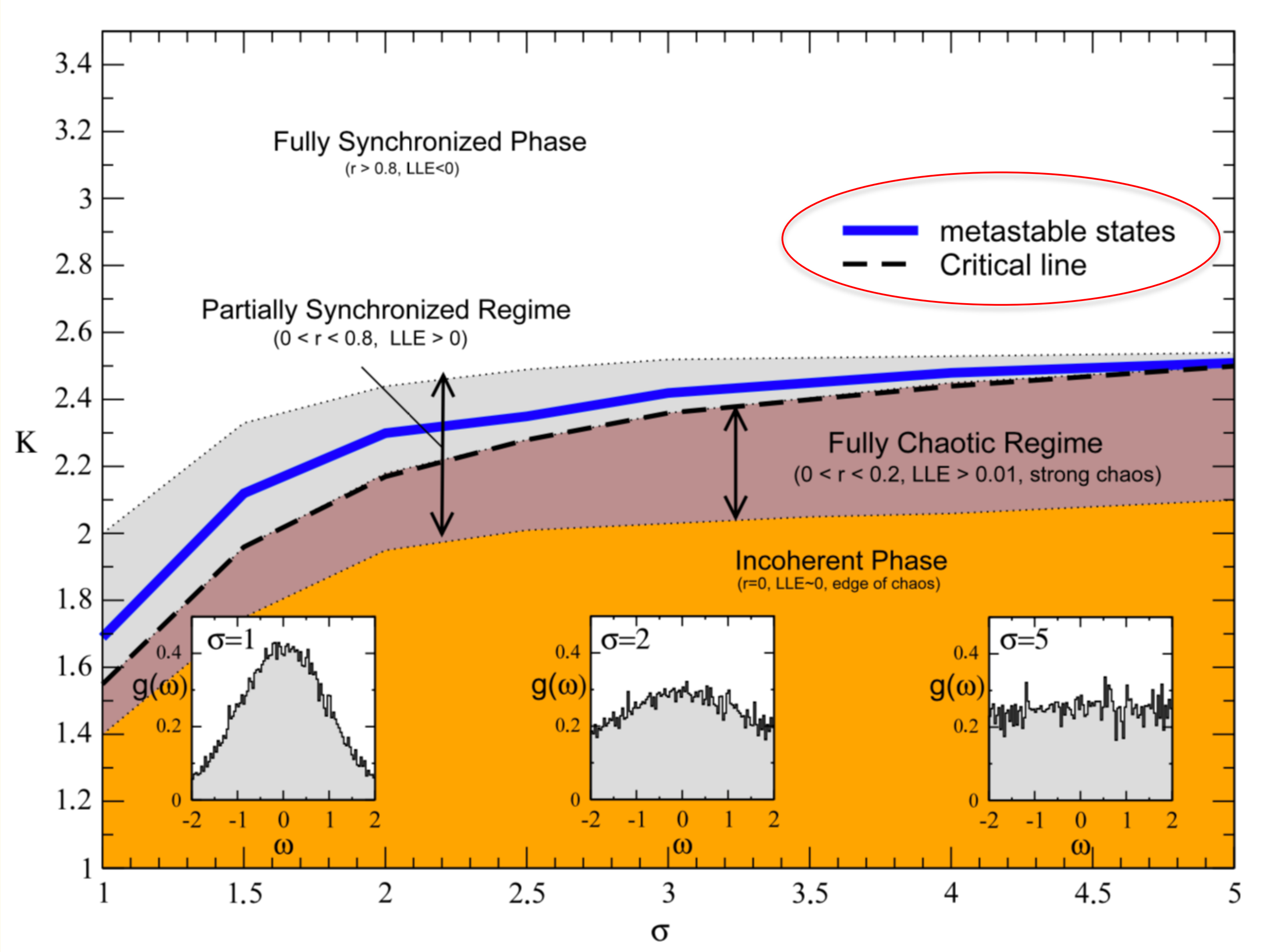

G Miritello, A Pluchino, A Rapisarda, Phase Transitions and Chaos in Long-Range Models of Coupled Oscillators
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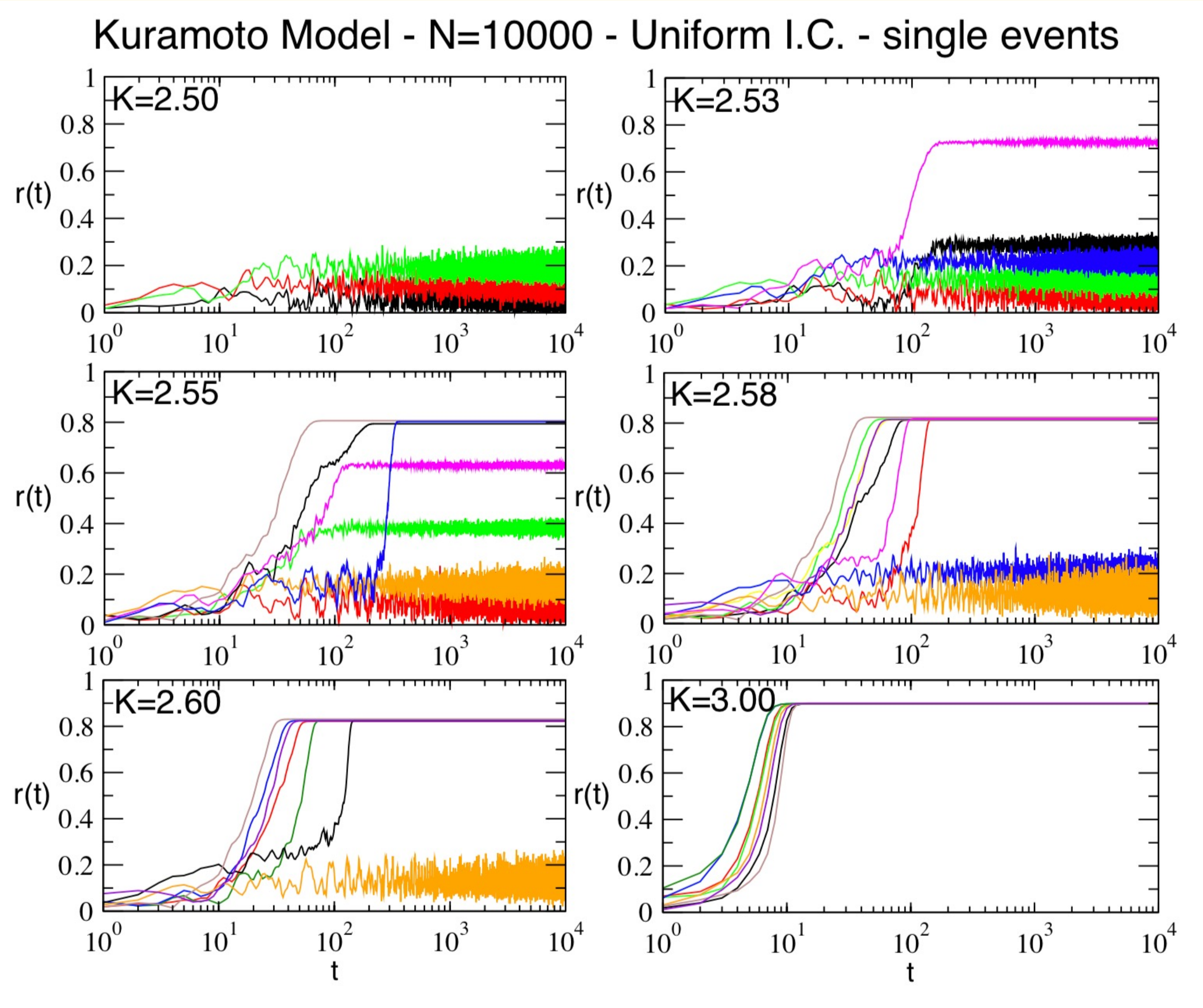
Metastable States in the Kuramoto Model



A Pluchino, A Rapisarda, Metastability in the Hamiltonian Mean Field model and Kuramoto model.
PHYSICA. A. vol. 365, pp. 184-189 (2006)

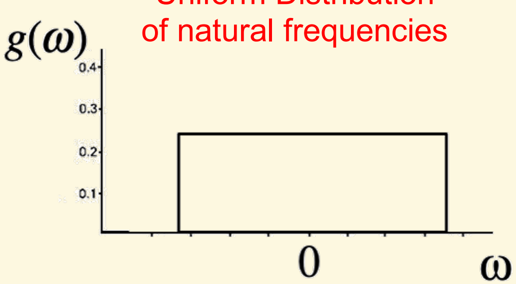
Central Limit Behavior in the Kuramoto Model at the "Edge of Chaos"

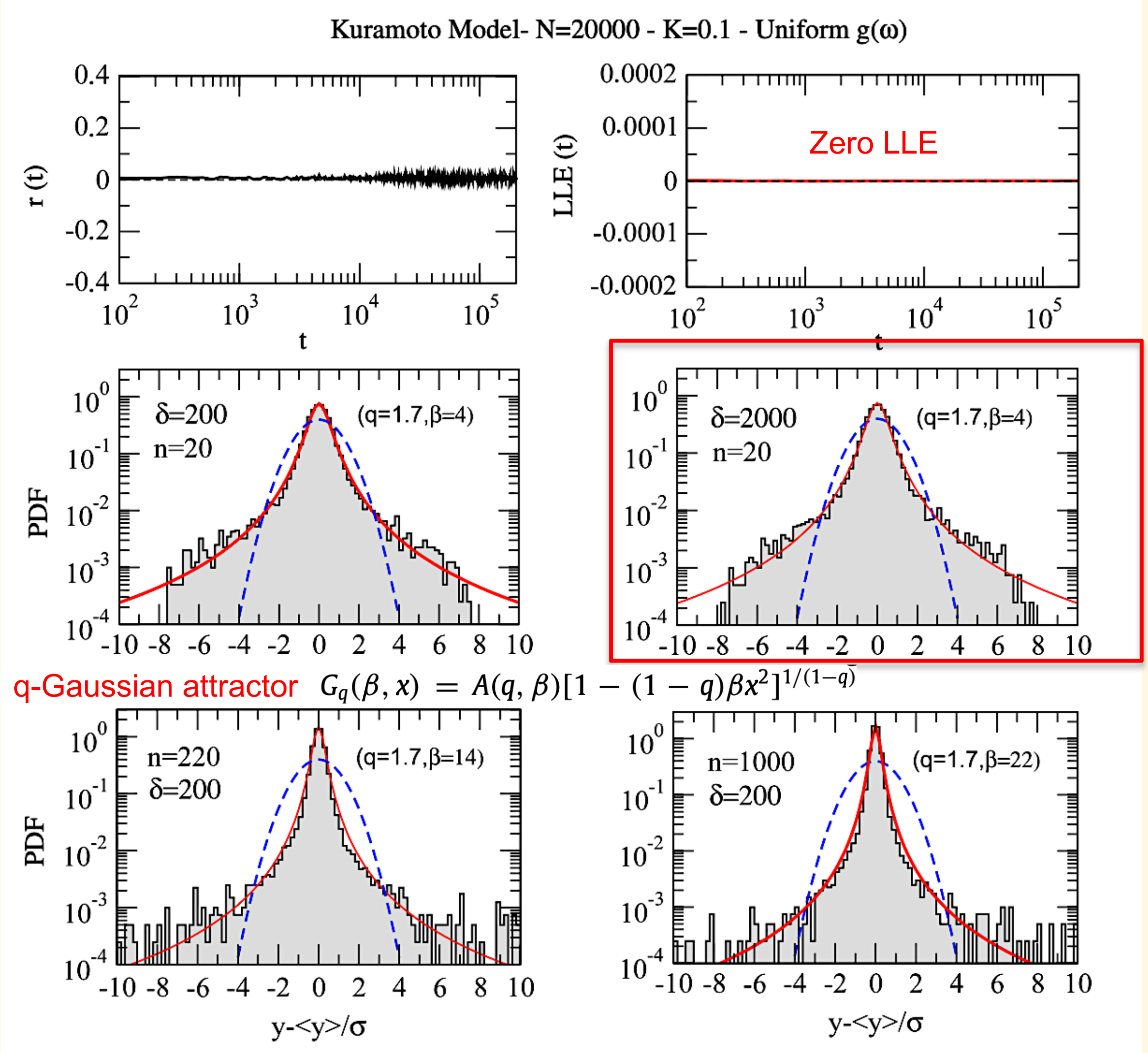
Rescaled sums obtained by picking out, for each oscillator, n values of the angle θ_i at fixed intervals of time δ along the deterministic time evolution:

$$y_i = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \theta_i(k\delta)$$

U. Tirnakli, C. Beck, C. Tsallis, Phys. Rev. E 75 (2007) 040106(R).

"Edge of Chaos" Regime





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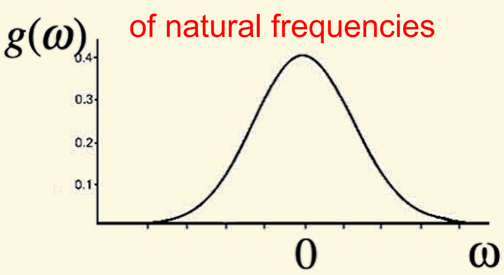

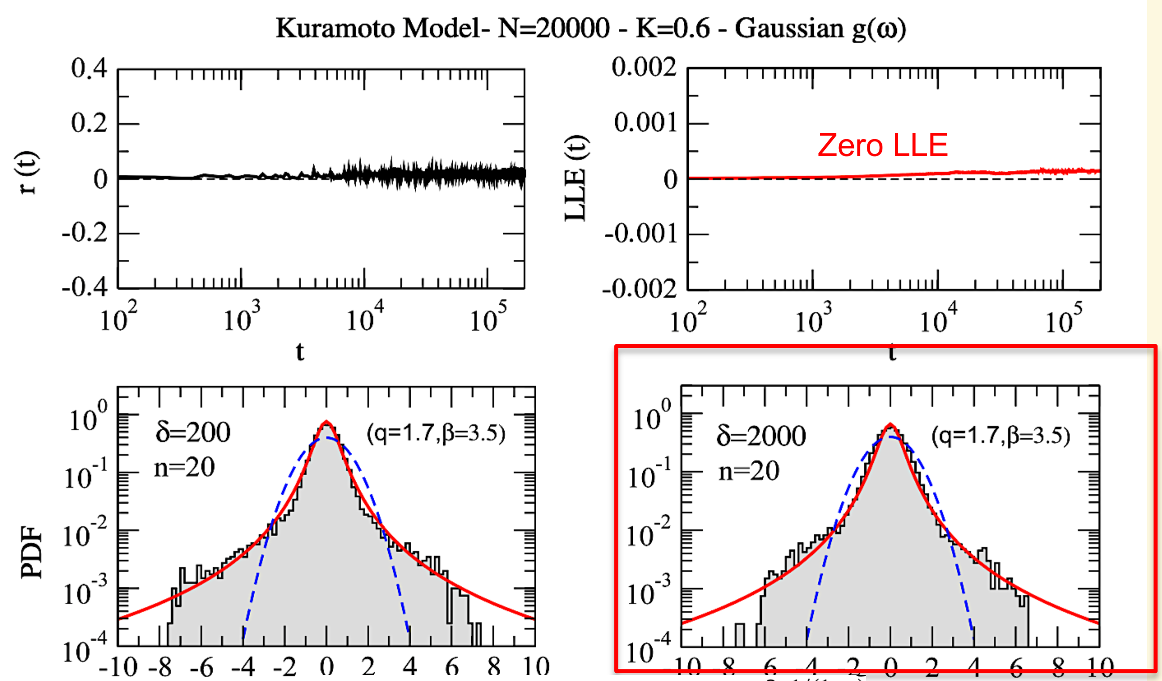

q-Gaussian attractor $G_q(\beta, x) = A(q, \beta)[1 - (1-q)\beta x^2]^{1/(1-q)}$

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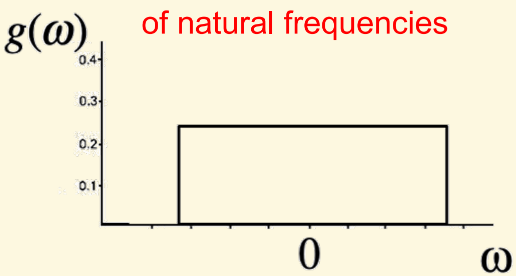
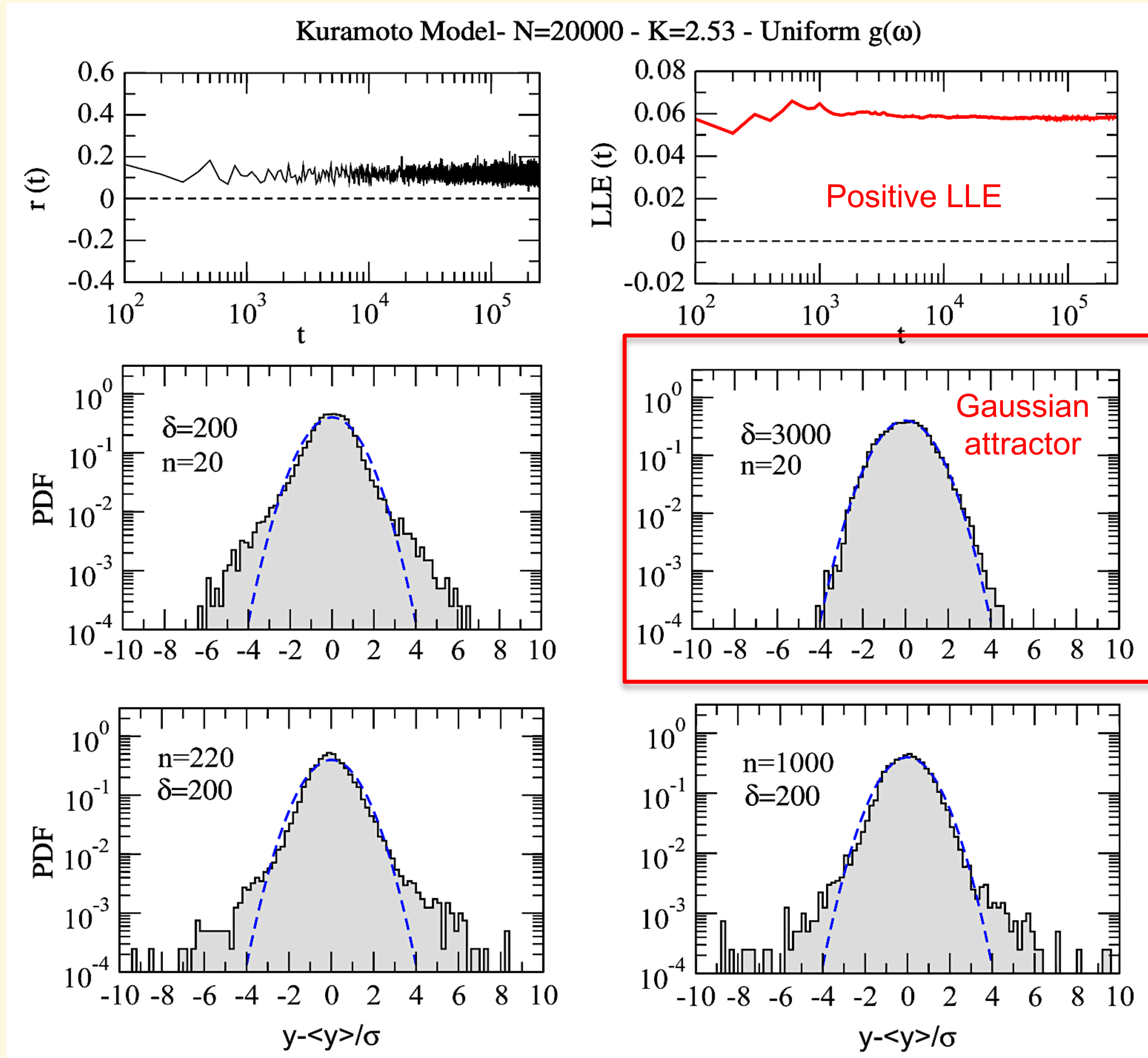
Fully Chaotic Regime

Uniform Distribution of natural frequencies





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Fully Chaotic Regime

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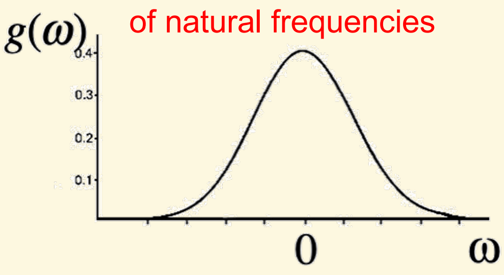

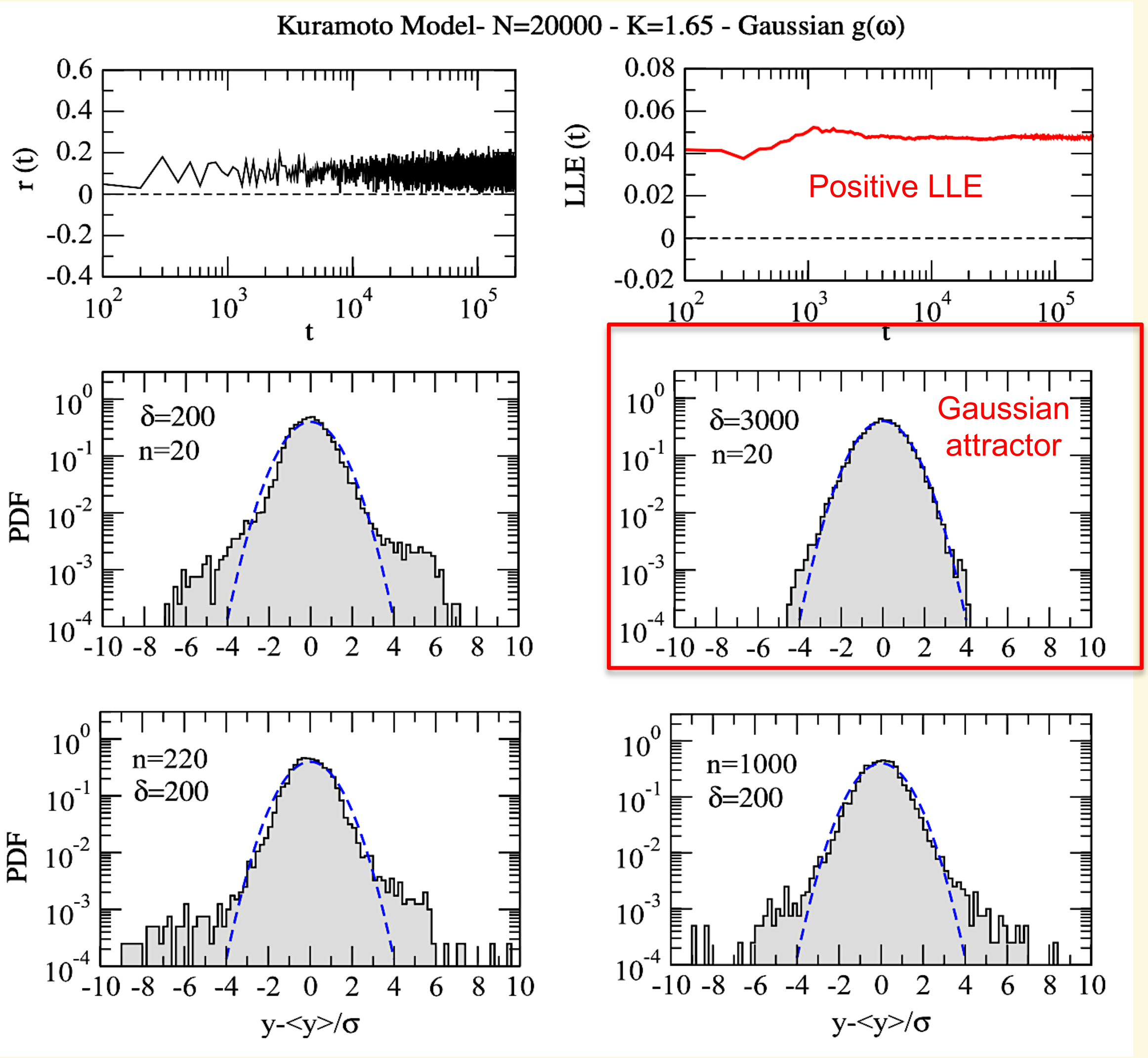

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2002

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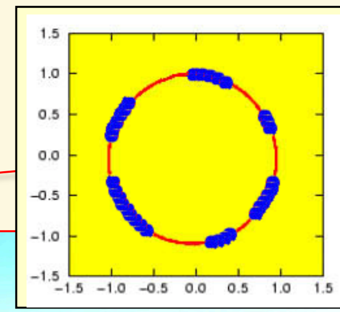
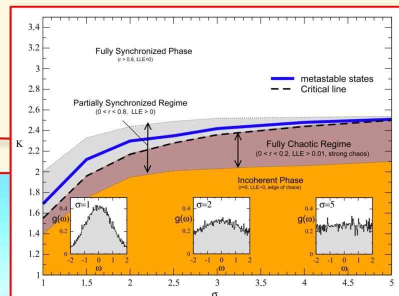
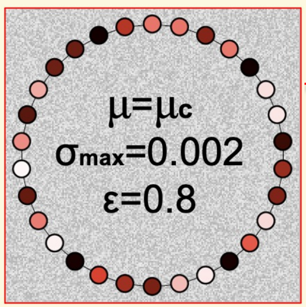


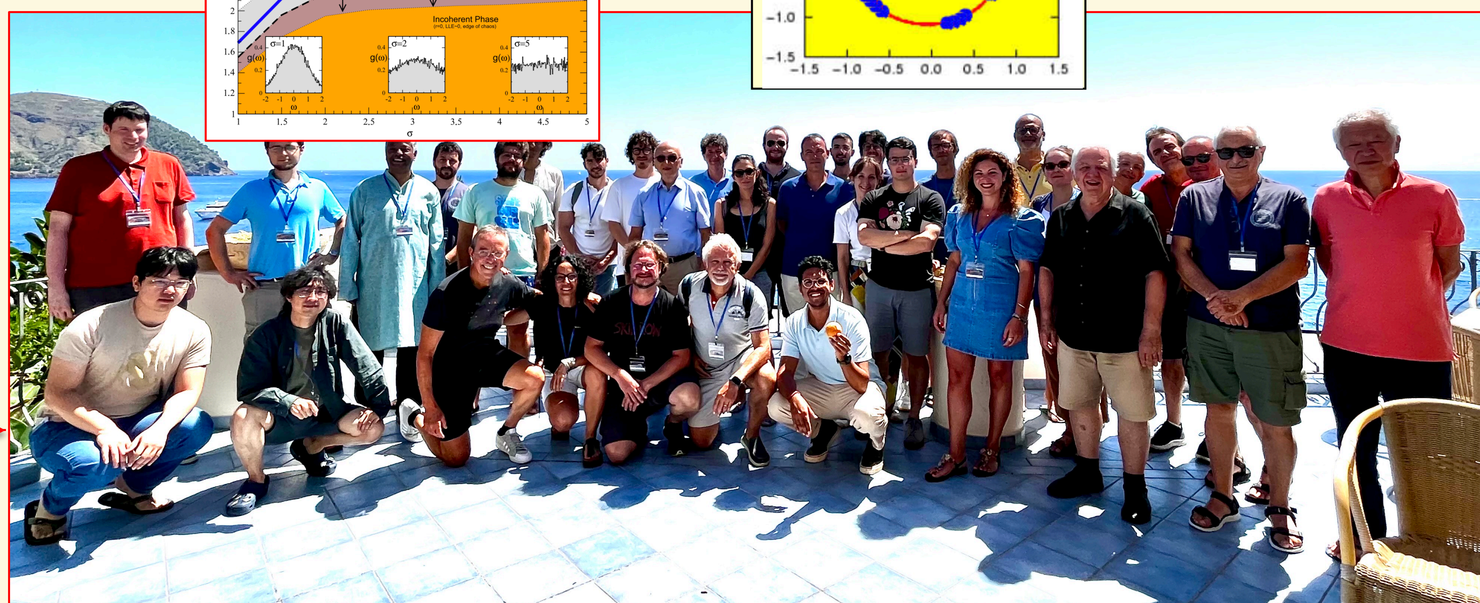
2009



2013



2023



Noise and Synchronization of Coupled Logistic Maps

Synchrony among coupled units has been extensively studied in the past decades providing important insights on the mechanisms that generate **emergent collective behaviors** in many complex systems.

In this context **coupled maps** have often been used as a theoretical model.

- Y. Kuramoto, "*Chemical Oscillations, Waves and Turbulence*" (Springer, New York, 1984)

- A. Pikovsky, M. Rosenblum and J. Kurths, "*Synchronization. A Universal Concept in Nonlinear Sciences*", (Cambridge 2001)

- S.H. Strogatz, "*Sync: The Emerging Science of Spontaneous Order*", (Hyperion Books, 2004)

- K. Kaneko , "*Simulating Physics with Coupled Map Lattices*" (World Scientific, Singapore, 1990)

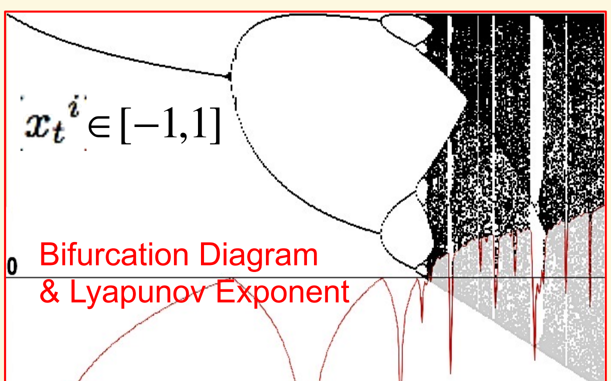
KANEKO CML MODEL: A 1D LATTICE of LOCALLY COUPLED LOGISTIC MAPS

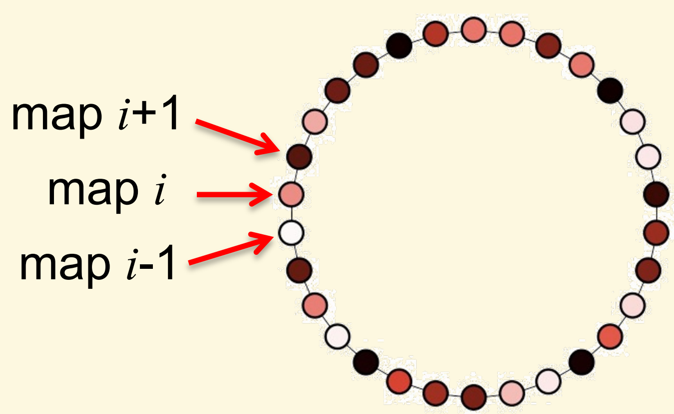
Single Logistic Map

$$f\left(x_t^i\right) = 1 - \mu\left(x_t^i\right)^2, \text{ with } \mu \in [0,2]$$

N Coupled Logistic Maps with periodic boundary conditions

$$x_{t+1}^i = (1-\epsilon) f\left({x_t}^i\right) + \frac{\epsilon}{2}\left[f\left({x_t}^{i-1}\right) + f\left({x_t}^{i+1}\right)\right]$$





Strenght of the local coupling

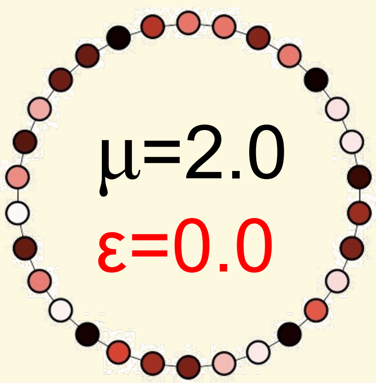
$$\epsilon \in [0,1]$$

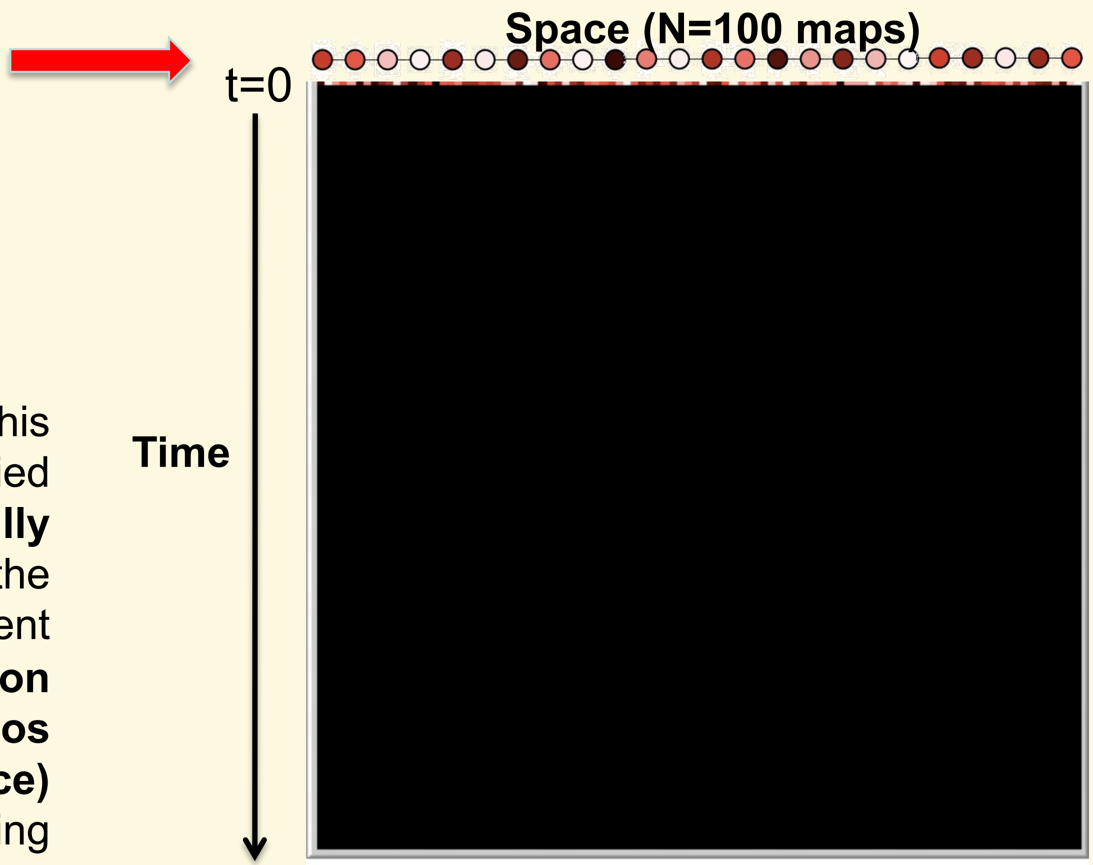
Different colors indicate different random initial conditions

Spatiotemporal chaos and synchronization patterns in the Coupled Map Lattice (CML) Model

K. Kaneko , "*Simulating Physics with Coupled Map Lattices*" (World Scientific, Singapore, 1990)

$$x_{t+1}^{i} = (1-\epsilon)\, f\left(x_t{}^{i}\right) + \frac{\epsilon}{2}\left[f\left(x_t{}^{i-1}\right) + f\left(x_t{}^{i+1}\right)\right]$$





In **absence of noise**, this model was extensively studied in particular **in the fully chaotic regime**, where the coupled maps show different **patterns of synchronization and spatiotemporal chaos (fully developed turbulence)** as function of the coupling strength ε.

Spatiotemporal chaos and synchronization patterns in the Coupled Map Lattice (CML) Model

K. Kaneko , “*Simulating Physics with Coupled Map Lattices*” (World Scientific, Singapore, 1990)

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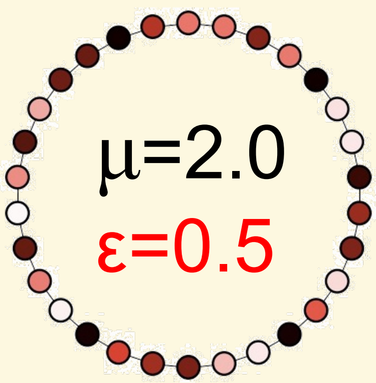

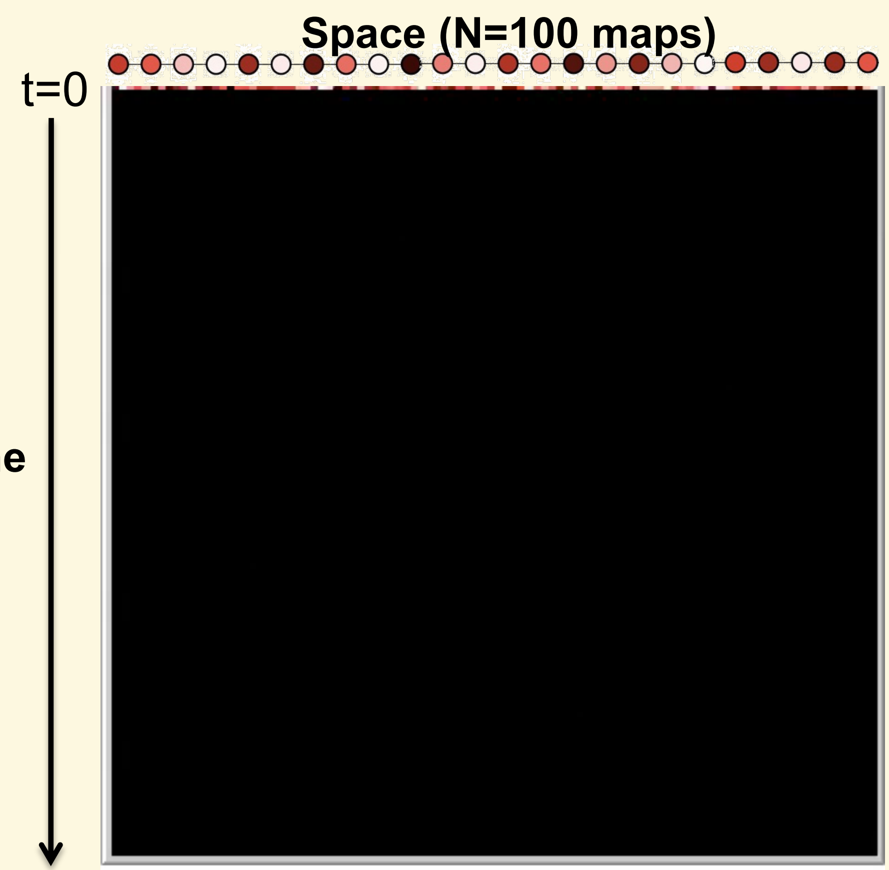

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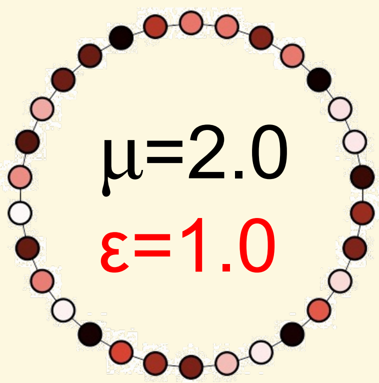
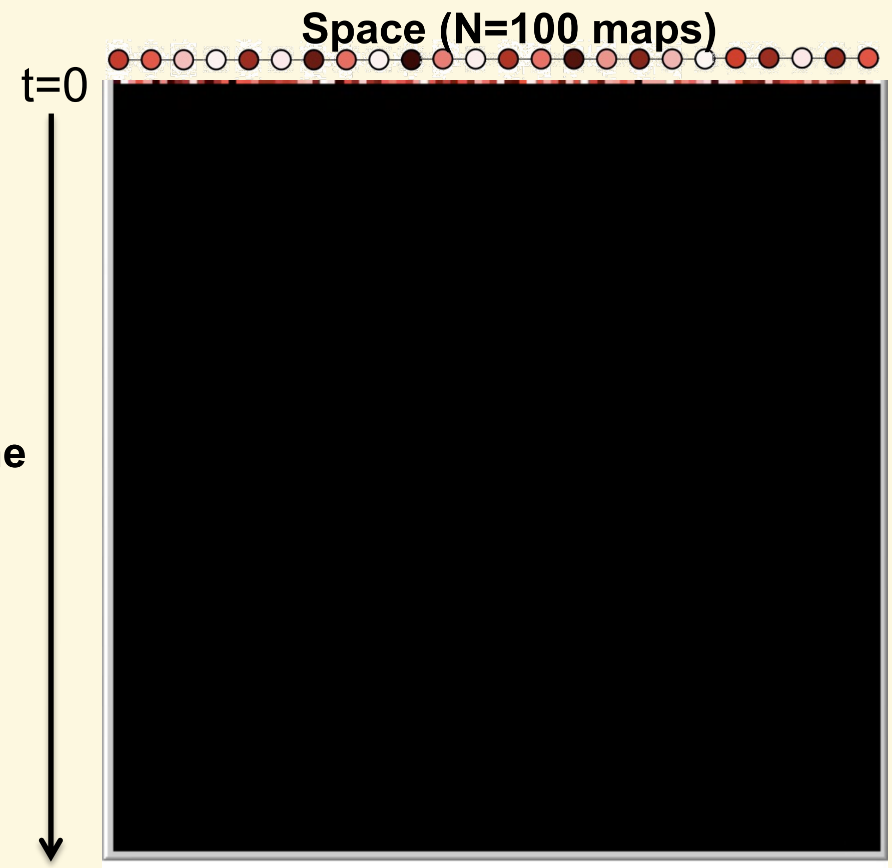


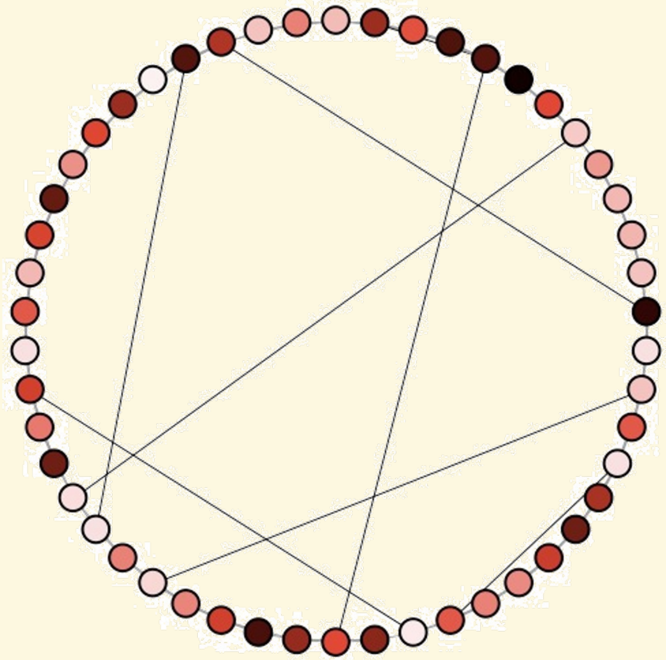


In **absence of noise**, this model was extensively studied in particular **in the fully chaotic regime**, where the coupled maps show different **patterns of synchronization and spatiotemporal chaos (fully developed turbulence)** as function of the coupling strength ε.

Inducing on-off intermittency in small-world networks of chaotic maps

C. Li and J. Fang, IEEE 0-7803-8834-8/05 (2005) 288 - 291 Vol. 1

It has been shown that **small-world topology** affects the behavior of the locally coupled logistic maps in the **fully chaotic** regime by introducing **long-range correlations** among maps. For a fixed strong coupling ε, when the **rewiring probability *p*** is slightly **less** than a critical value (0.29), the synchronous chaotic state is no longer stable and **on-off intermittency** appears.

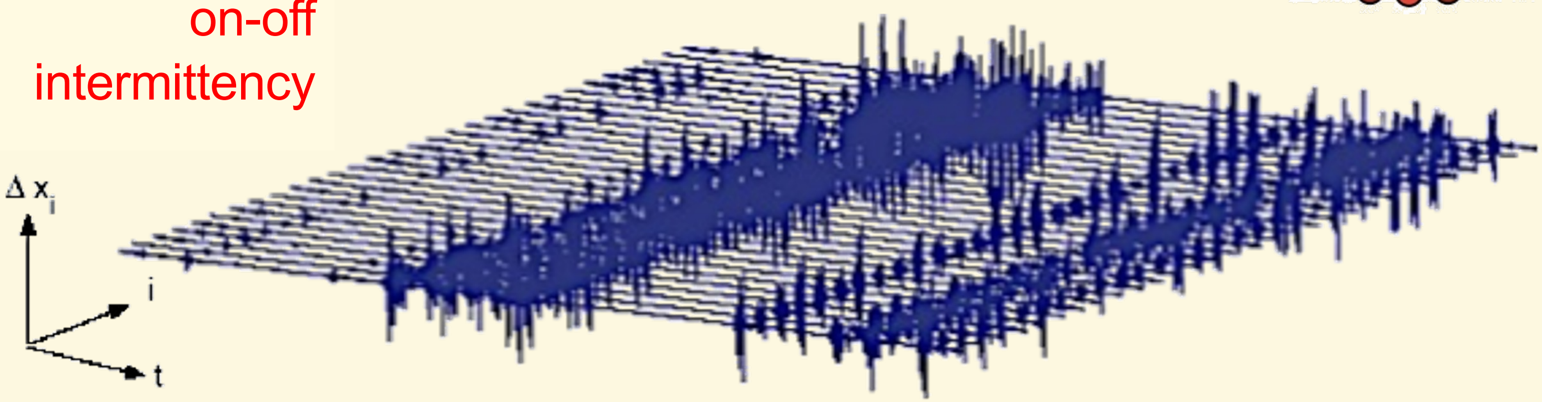


μ=1.9

ε=0.6

***p* = 0.27**

on-off intermittency



Noise induced correlations in a lattice of logistic maps at the edge of chaos

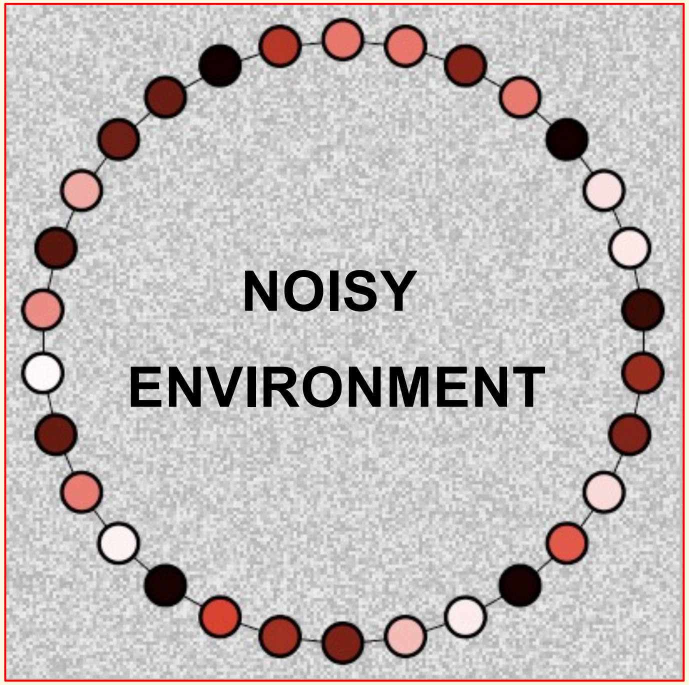
Our idea is to induce **long-range correlations and intermittency** in the system using local coupling only but **embedding the maps in a common noisy environment:**

$$x_{t+1}^i = (1-\epsilon)\, f\left(x_t{}^i\right) + \frac{\epsilon}{2}\left[f\left(x_t{}^{i-1}\right) + f\left(x_t{}^{i+1}\right)\right] + \sigma(t)$$

$f\left(x_t^i\right)$ taken in module 1 with sign

the additive noise is a random variable uniformly extracted in the interval

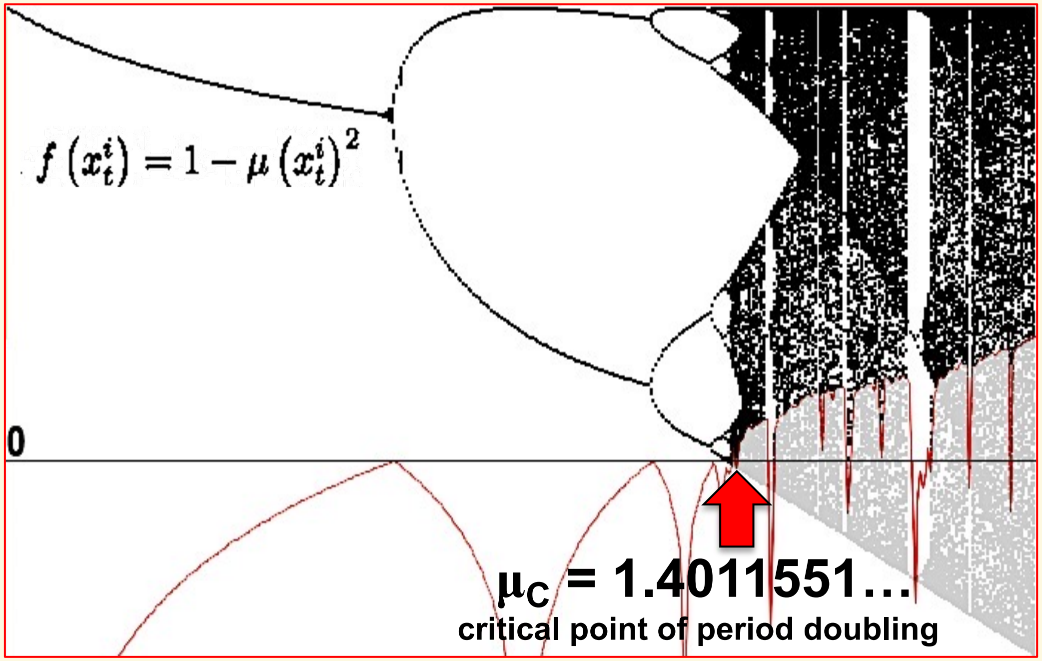
$\sigma(t) \in [0, \sigma_{max}]$



A Pluchino, A Rapisarda, C Tsallis
Noise, synchrony, and correlations at the edge of chaos.
Physical Review E vol. 87 (2): 022910 (2013)

Noise induced correlations in a lattice of logistic maps at the edge of chaos

At variance with previous studies on coupled logistic maps we also consider them not in the chaotic regime but **at the edge of chaos**, where the *Lyapunov exponent is vanishing*:



Many biological complex systems operate frequently both **at the edge of chaos and in a noisy environment**. Therefore studying the effect of a weak noise in this kind of coupled systems could be relevant in order to **understand the way in which interacting units behave in real complex systems,** like for example living cells.

See e.g.: - D. Stokic, R. Hanel, S. Thurner, Phys. Rev. E. 77, 061917 (2008)

- R. Hanel, M. Pochacker, M. Scholling, S.Thurner, Plos One bf 7, e36679 (2012)

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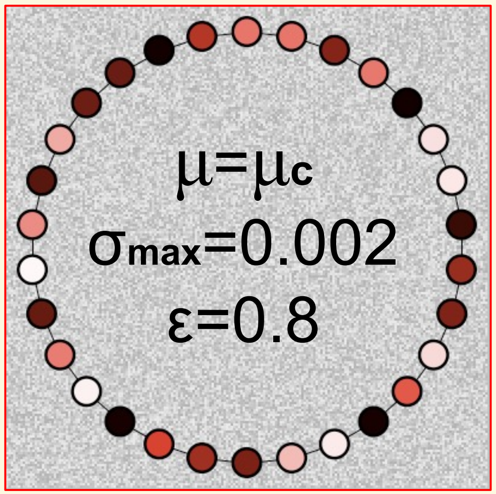
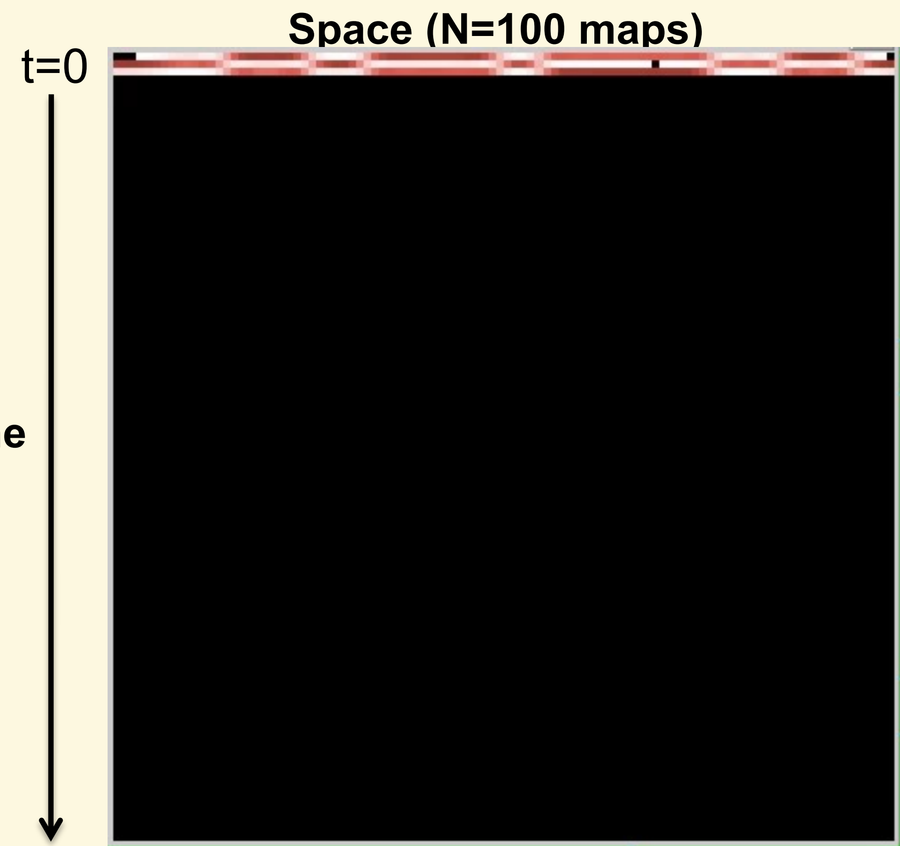
Noise induced correlations in a lattice of logistic maps at the edge of chaos

$$x_{t+1}^{i} = (1-\epsilon)\, f\left({x_t}^{i}\right) + \frac{\epsilon}{2}\left[f\left({x_t}^{i-1}\right) + f\left({x_t}^{i+1}\right)\right] + \sigma(t)$$





The addition of a **small level of noise** induces evident **spatiotemporal correlations** to the lattice of logistic maps at the edge of chaos, in presence of **strong coupling**.

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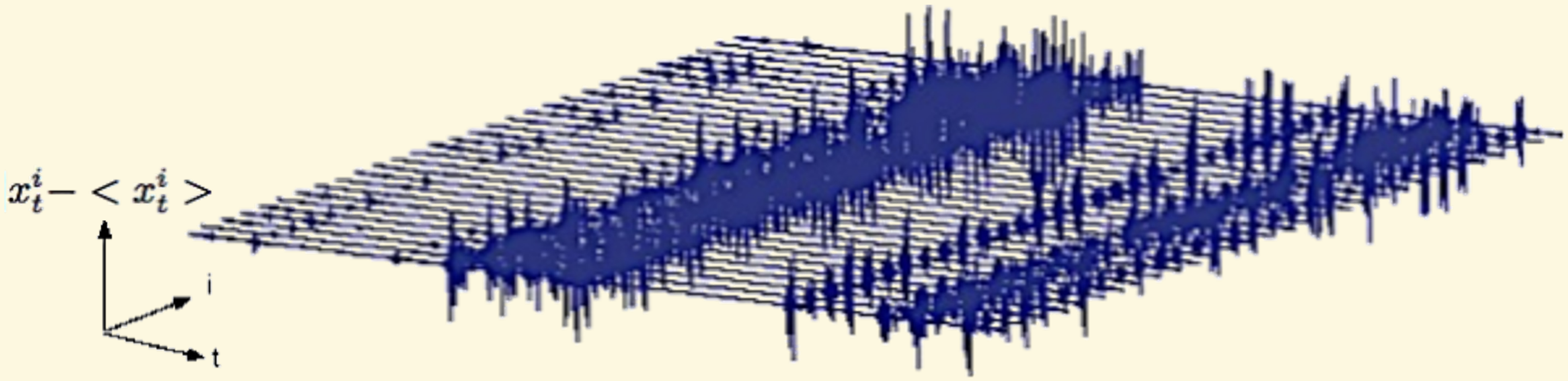
Noise induced correlations in a lattice of logistic maps at the edge of chaos

In order to study these **correlations** we subtract the synchronized component and **keep the desynchronized part** of each map, considering, at every time step, the difference between the average and the single map value. Then we further consider the **average of the absolute values** of these differences over the whole system in order to measure the **distance from the synchronization regime at time t** with only one variable:

$$d_t = \frac{1}{N}\Sigma_{i=1}^{N}|x_t^i - < x_t^i > |$$

If all maps are trapped in some **synchronized pattern** then this quantity remains close to zero, otherwise **oscillations** are found.



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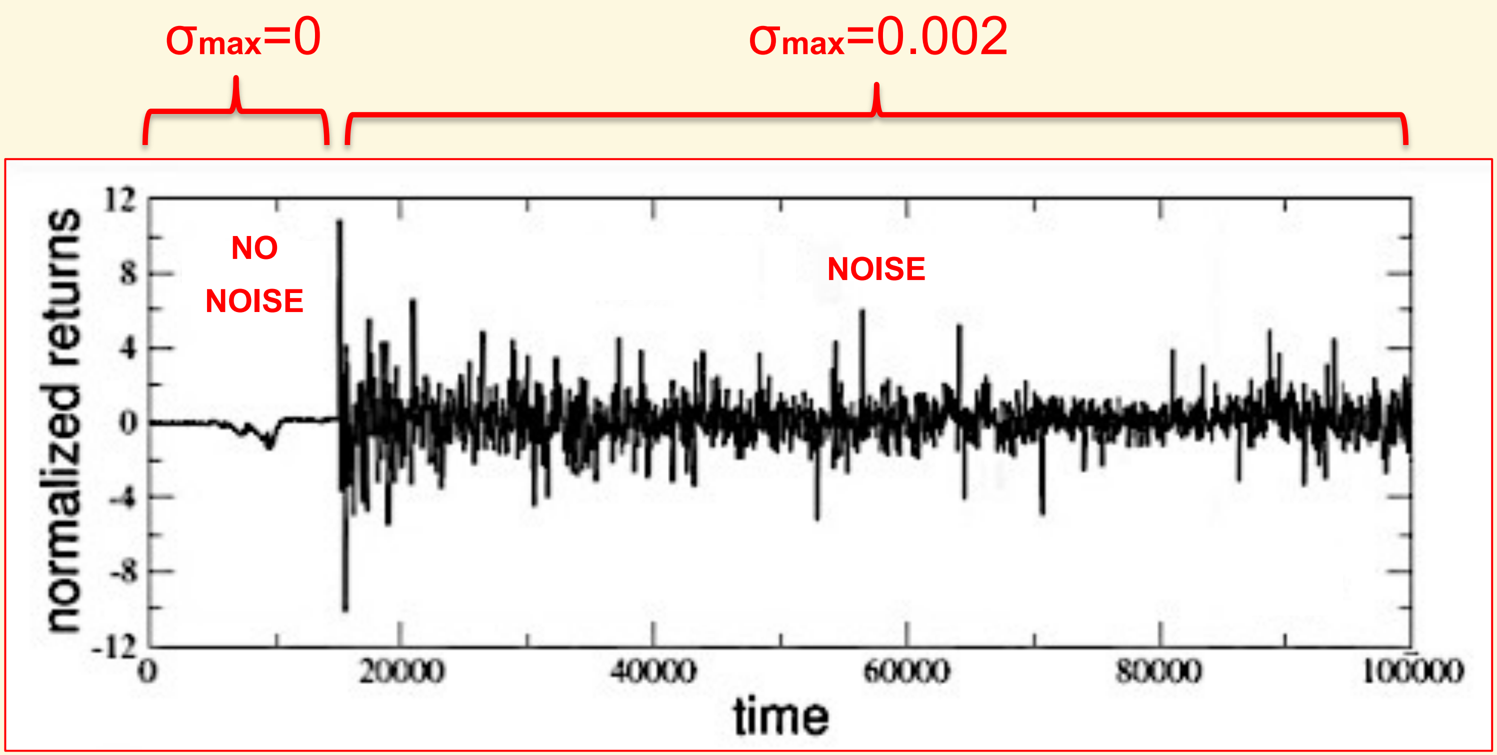
As commonly used in turbulence or in finance, we analyze these oscillations by considering the **two-time returns** Δd_t with an **interval of τ time steps**, defined as:

$$\Delta d_t = d_{t+\tau} - d_t$$

- S.Rizzo, A.Rapisarda, "*Application of superstatistics to atmospheric turbulence*" in Complexity, Metastability and Nonextensivity, World Scientific, Singapore (2005) 39
- J. Ludescher, C. Tsallis and A. Bunde, Europhys. Letters 95, 68002 (2011)

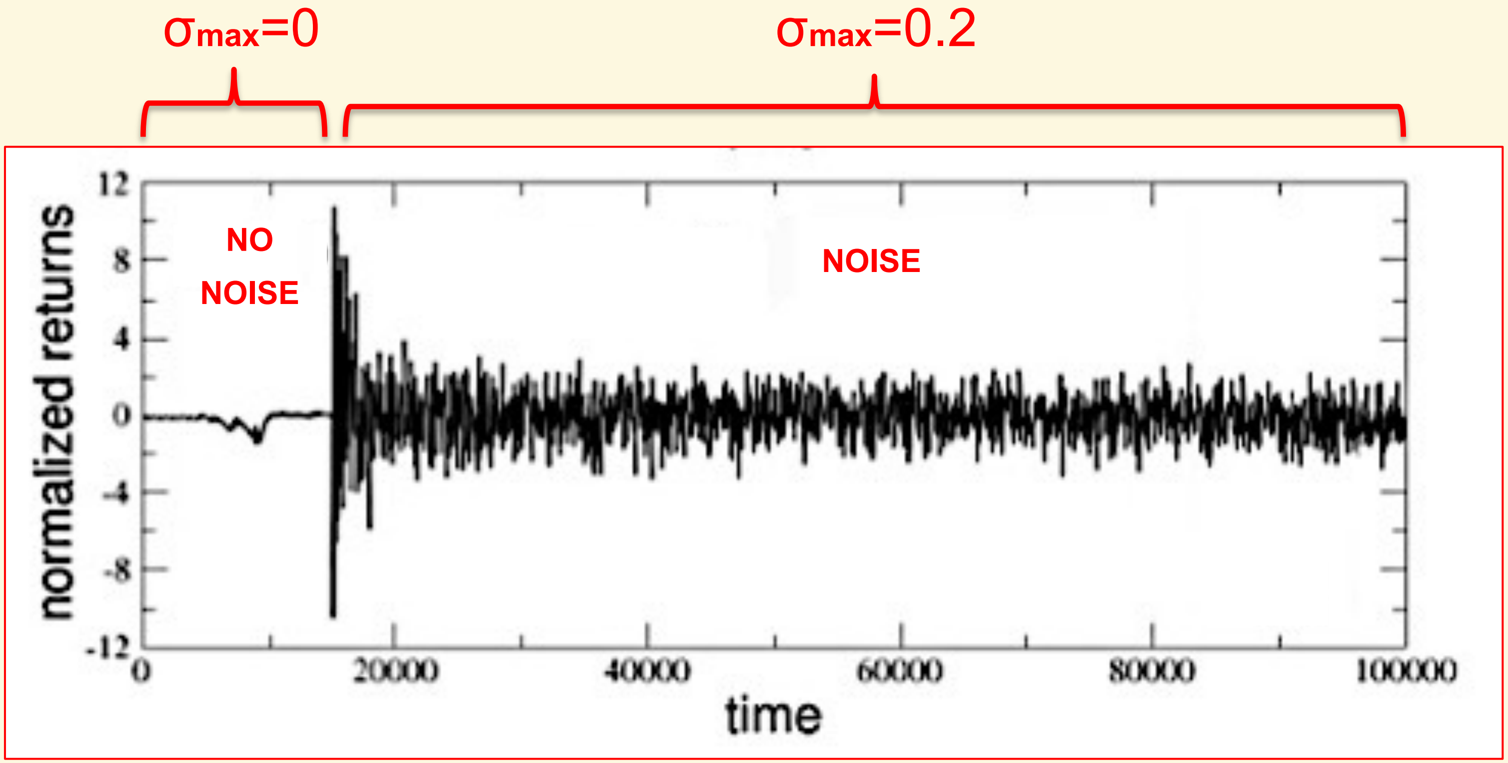
Time evolution of the two-time returns in presence of weak noise

Effect of noise in the time evolution of returns (normalized to the standard deviation of the overall sequence) for the case **$N = 100$, $\mu = \mu_c = 1.4011551...$, $\varepsilon = 0.8$ and $\tau = 32$ time steps.** During the first 15.000 time steps at **zero noise** (**$\sigma_{max} = 0$**) the maps remain synchronized due to the strong coupling. At time t = 15000 we **switch on the noise**, with **$\sigma_{max} = 0.002$ (weak noise)**: a clear **intermittent behavior** appears.



Time evolution of the two-time returns in presence of weak noise

The intermittent behavior **disappears** if we repeat the same simulation but with **σ_{max} = 0.2**, i.e. in presence of **strong noise.** In this case only **Gaussian fluctuations** are observed.

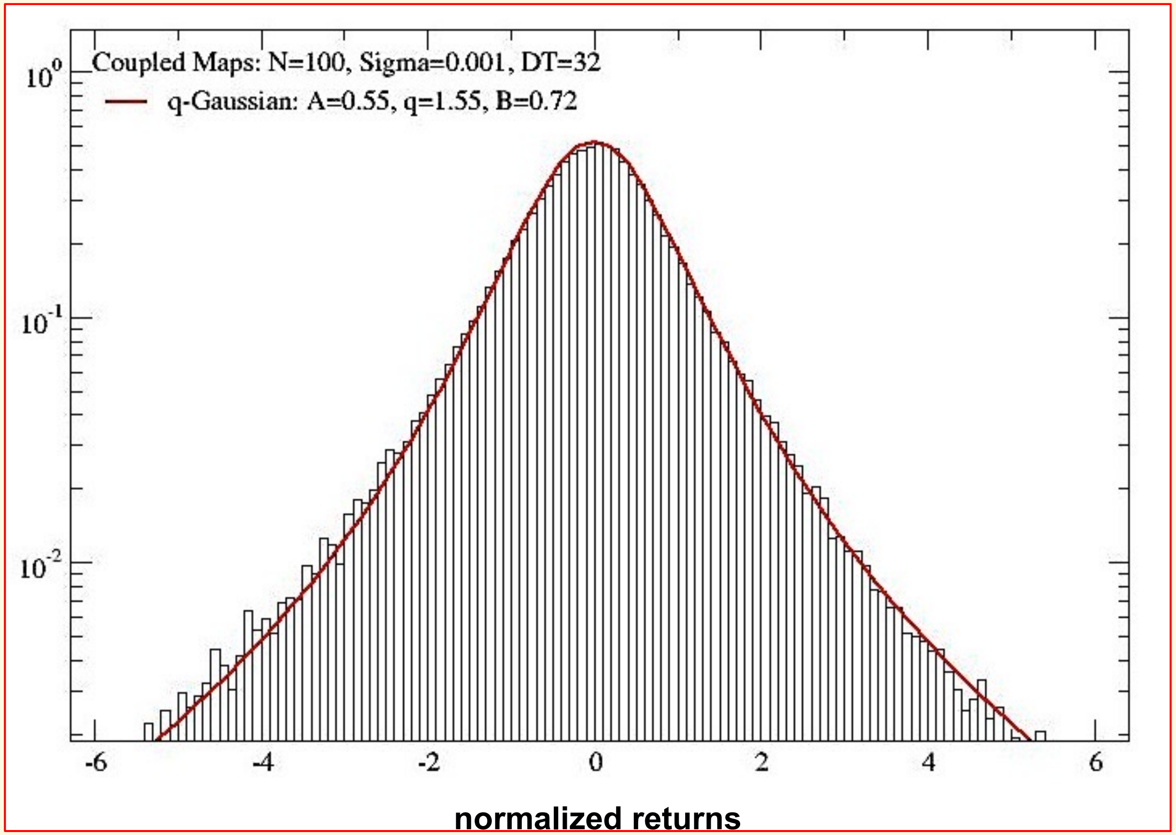


PDFs of normalized returns for increasing values of noise

To better appreciate the transition from the intermittent to the Gaussian behavior, we plot the **probability density function (Pdf) of the normalized returns** for several **increasing values of noise.** Fat tails in the Pdfs are clearly visible only when σ_{max} < 0.05 and can be nicely reproduced by ***q*-Gaussian curves** with decreasing values of the entropic index:

σ_{max}=0.001

q=1.55



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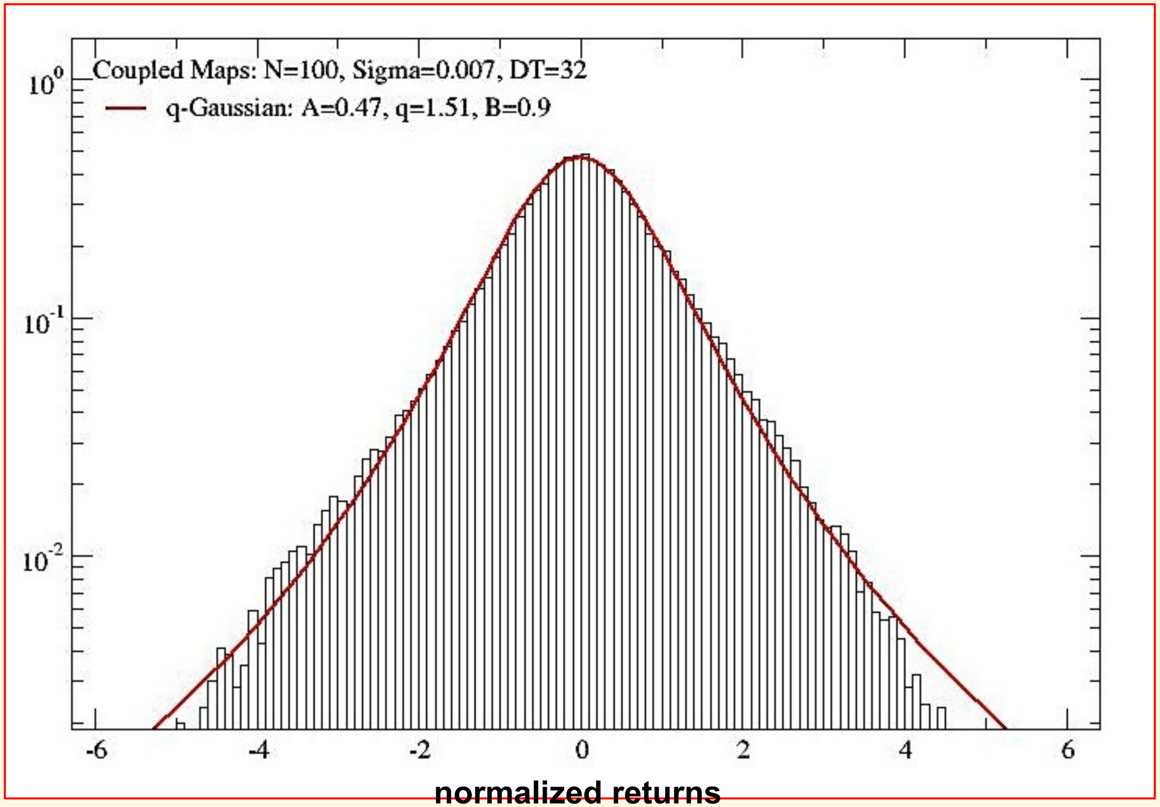
σ_{max}=0.007

q=1.51



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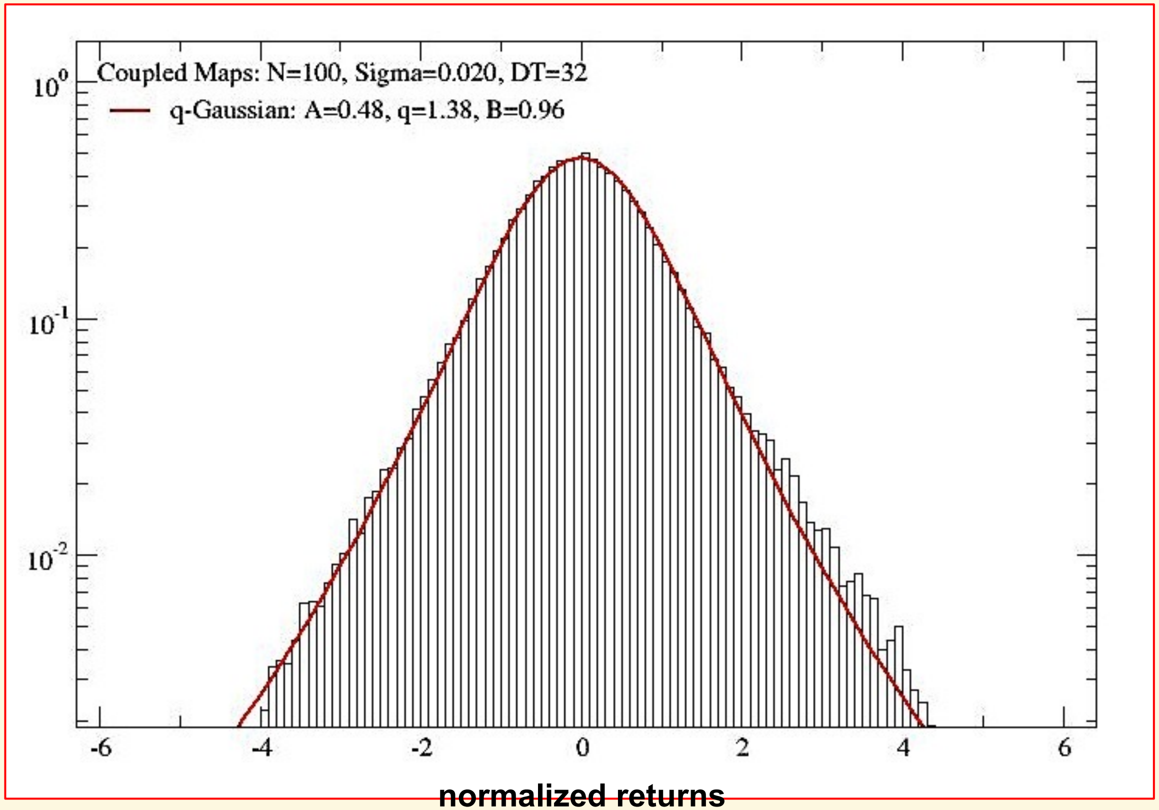
σ_{max}=0.02

q=1.38



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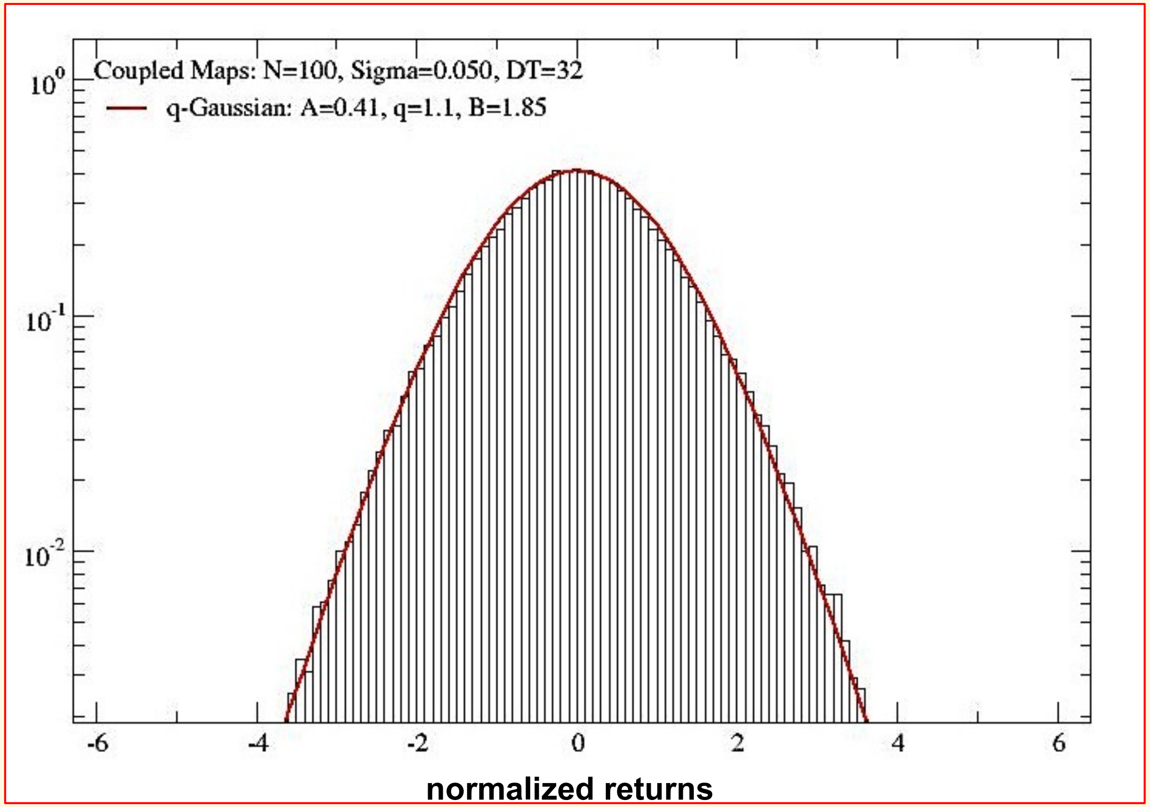
σ_{max}=0.05

q=1.10



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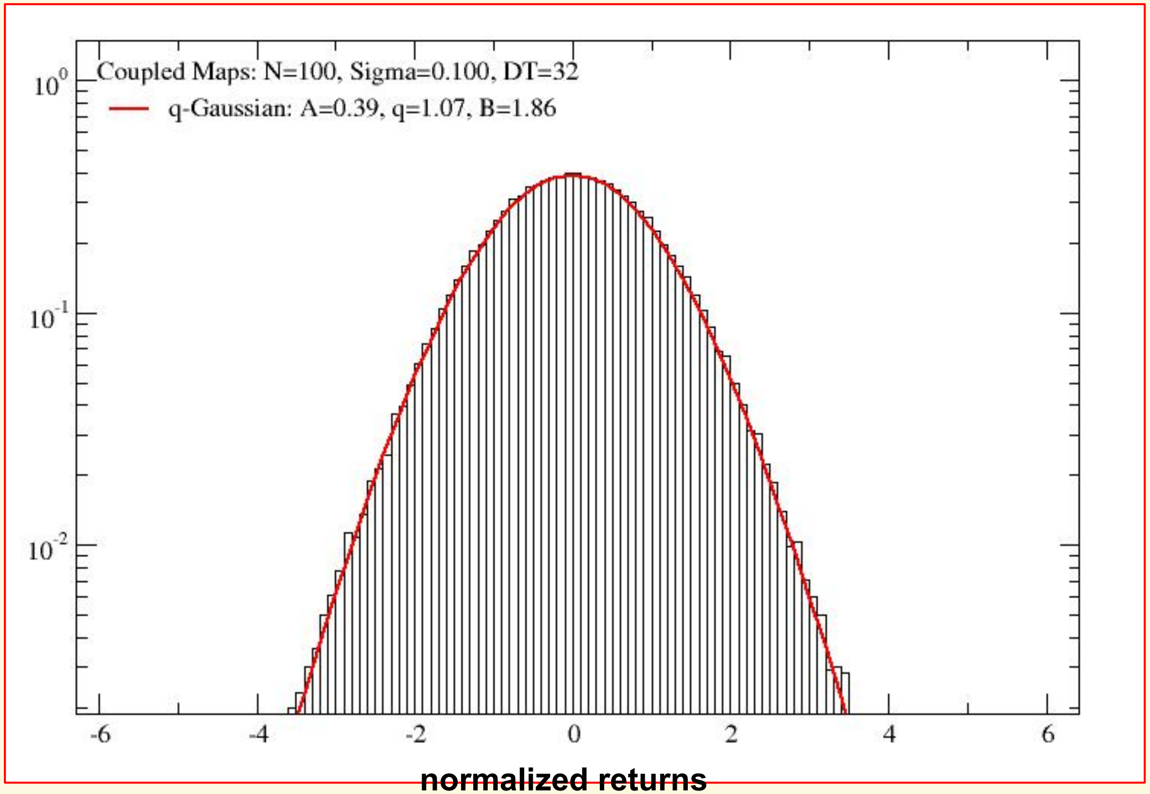
σ_{max}=0.10

q=1.07



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σ_{max}=0.20

q=1.01

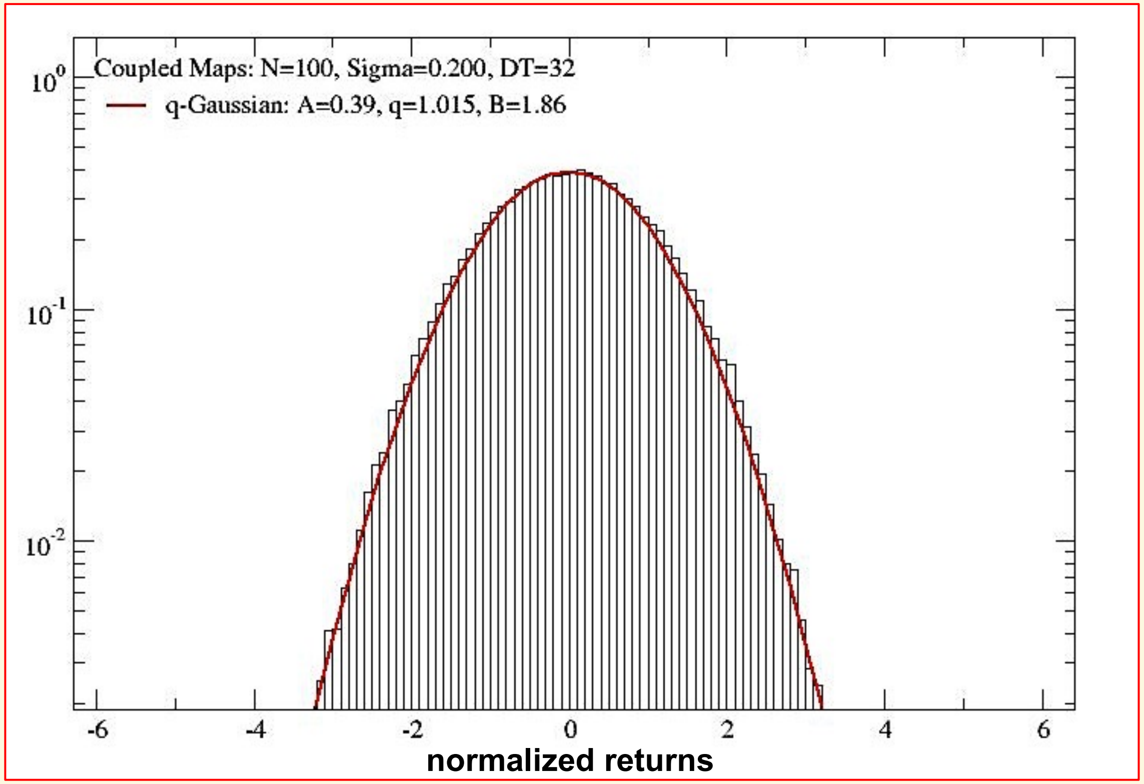

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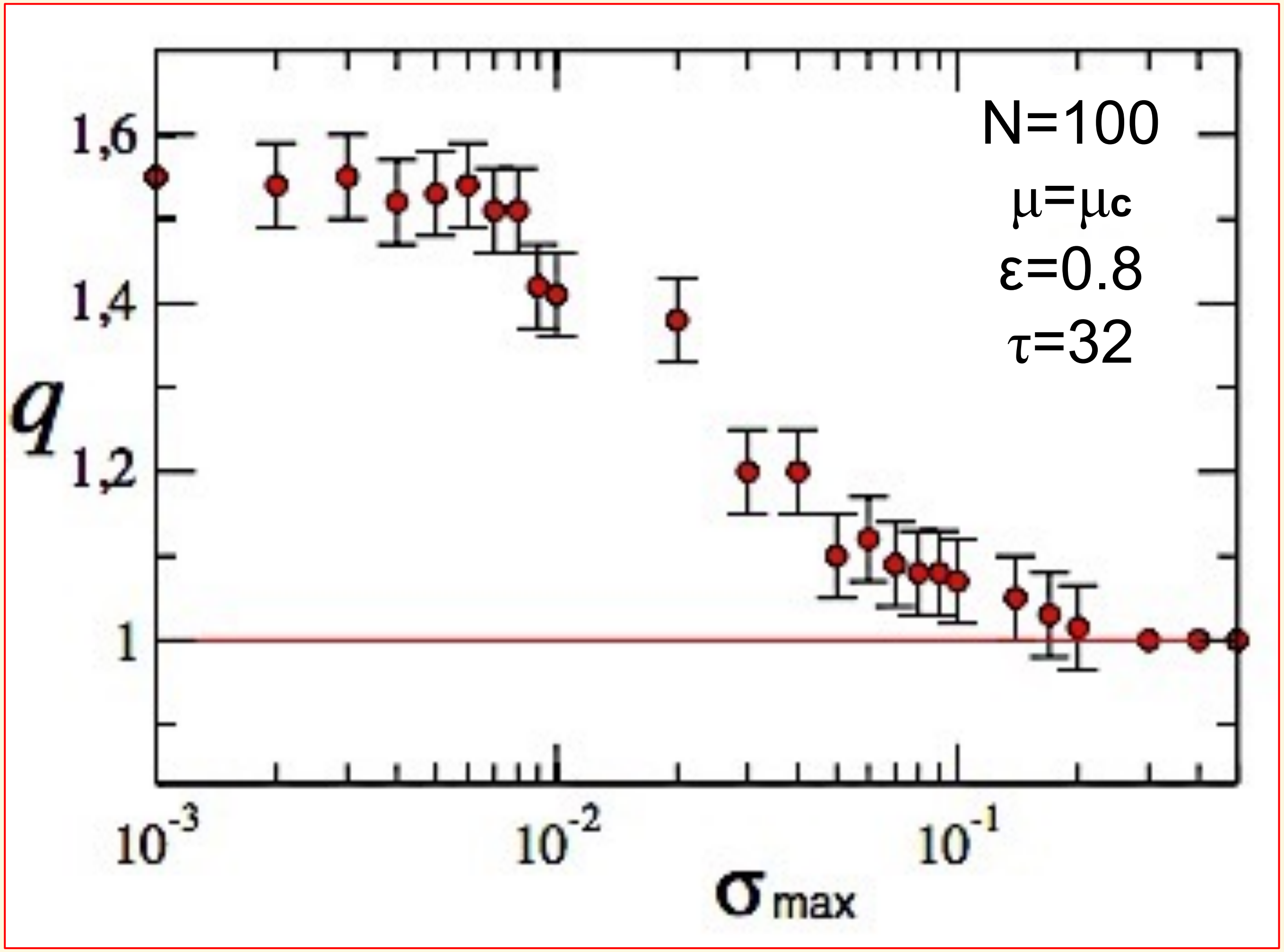
Diagram of q versus σ



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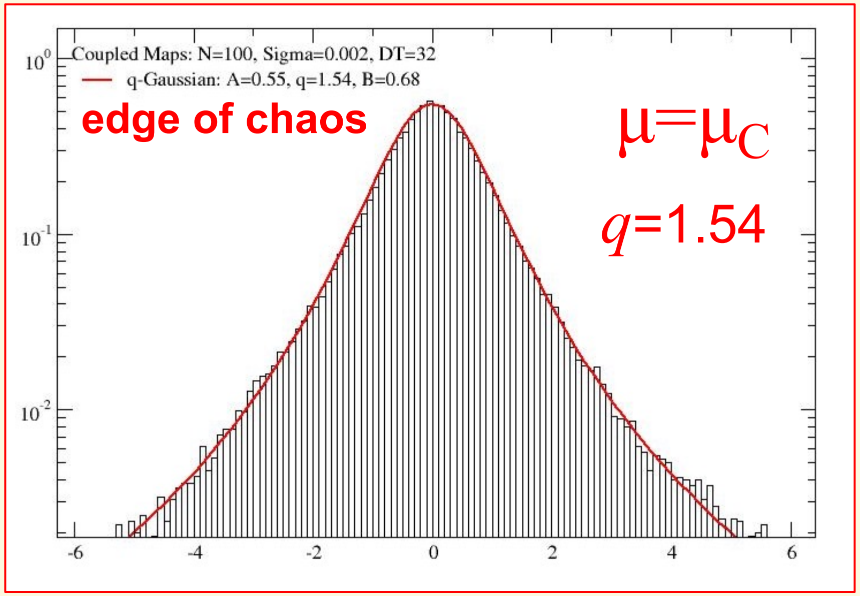
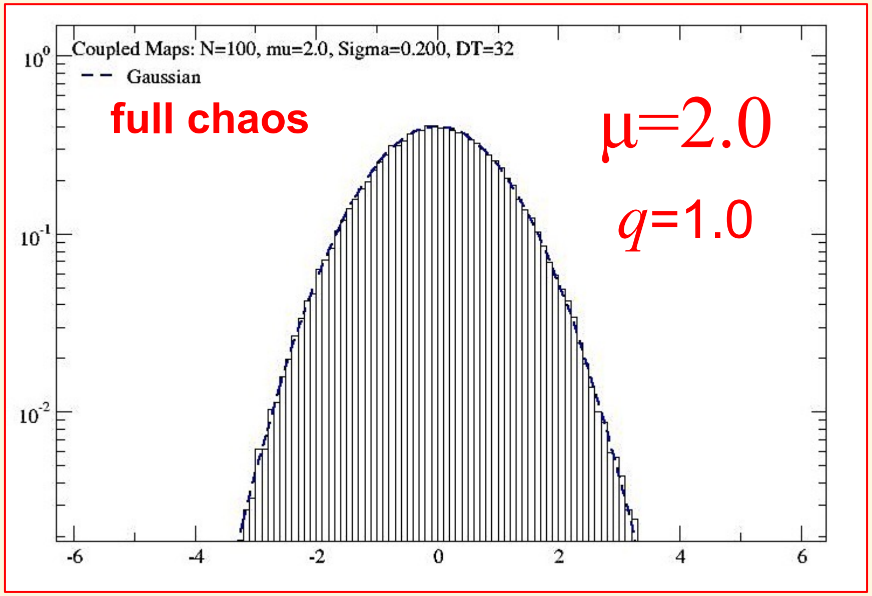
Gaussian behavior of returns in the fully chaotic regime

The edge of chaos condition is strictly necessary for the emergence of intermittency and strong correlations in presence of a small level of noise. In fact, if we consider the maps in the **fully chaotic regime**, i.e. with $\mu = 2$ instead of $\mu = \mu_c$, and leaving all the other parameters unchanged, we obtain a **Gaussian Pdf** of returns.

$N=100$, $\sigma_{max}=0.002$, $\varepsilon=0.8$, $\tau=32$





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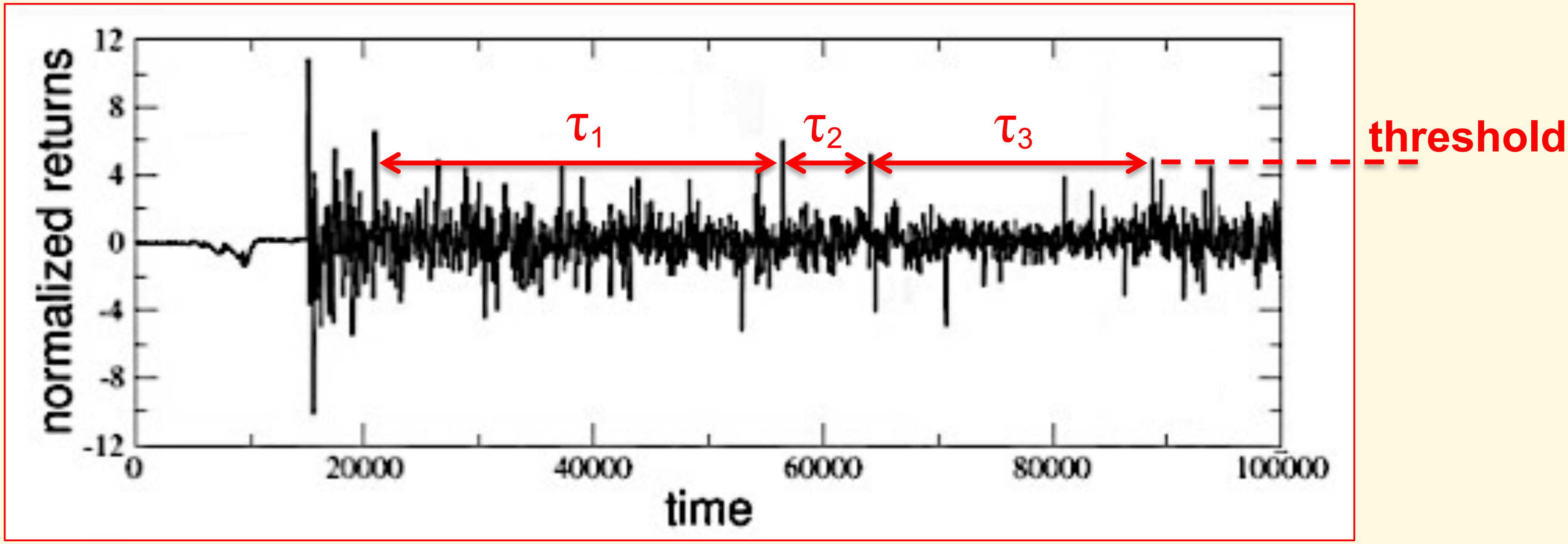
Analysis of the interoccurence times

Long-term correlations in a system typically yield powerlaw asymptotic behaviors in various physically relevant properties. In studies of **financial markets***, it was recently observed **power-law decays** in the so-called '**interoccurrence times**' between sub sequential peaks in the fluctuating time series of returns. If we fix a given **threshold**, the sequence of the interoccurrence times (τ_i) results to be well defined and it is then possible to study its Pdf for our system of coupled maps at the edge of chaos.

* M.I. Bogachev and A. Bunde, Phys. Rev. E 78, 036114 (2008)

N=100, σ_{max}=0.002, $\mu=\mu_c$, ε=0.8, τ=32



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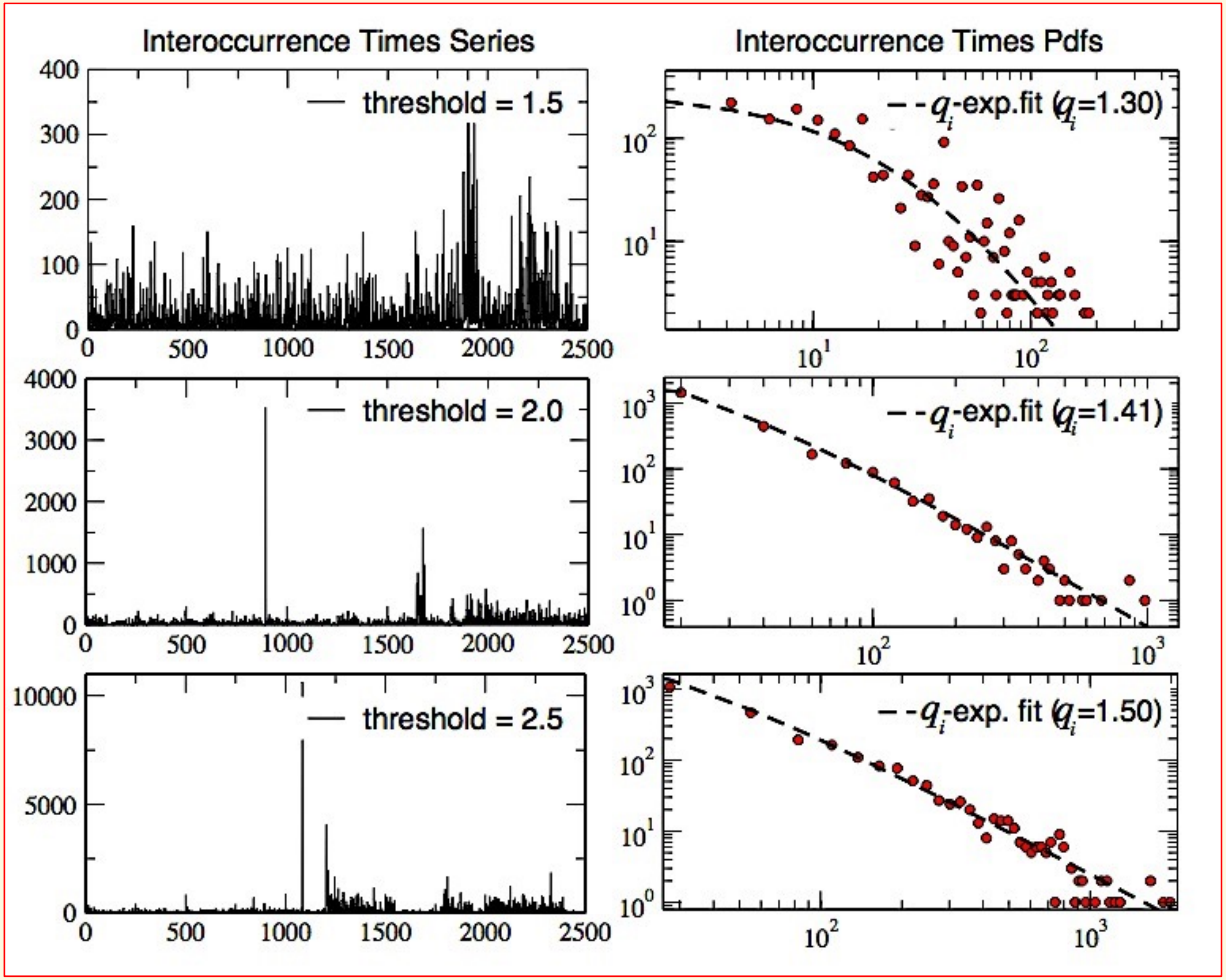
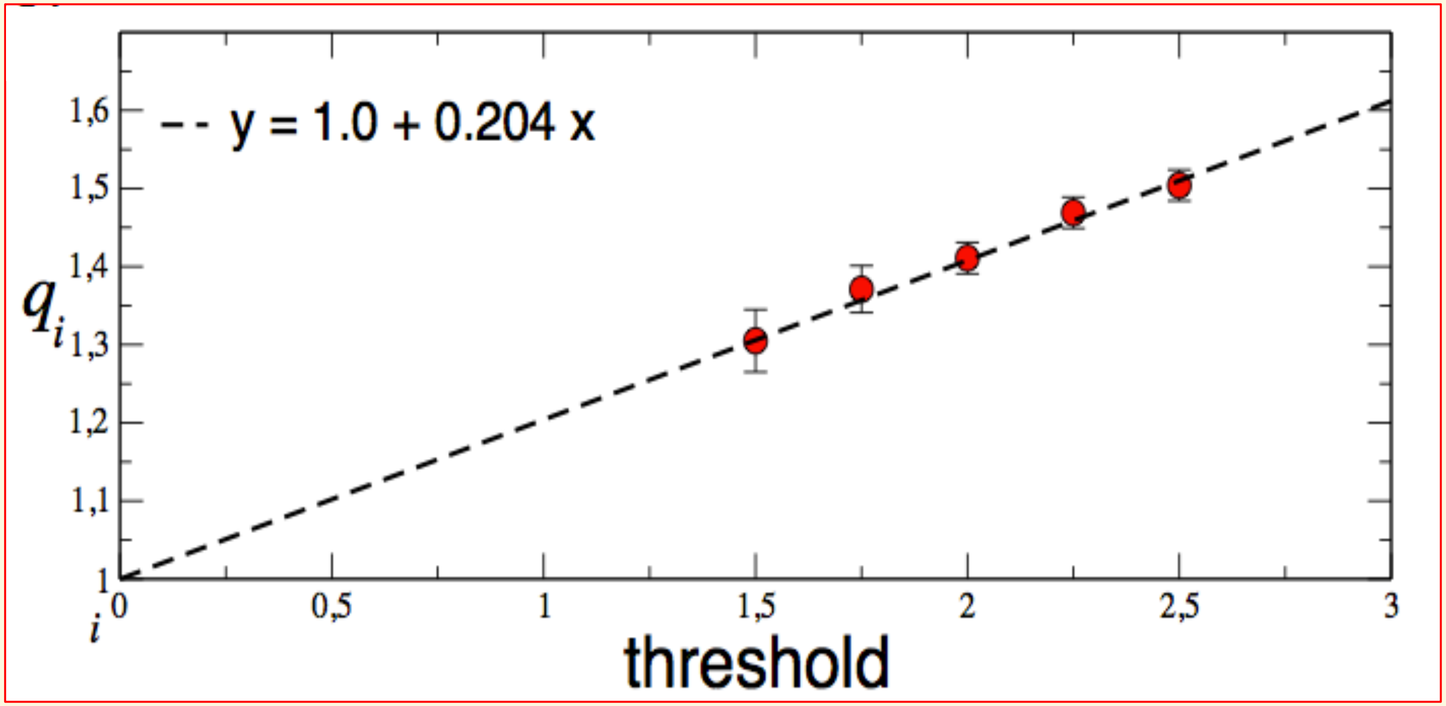
N=100, σ_{max}=0.002, μ=μc, ε=0.8, τ=32



Analysis of the interoccurence times

This can be considered as a **further footprint of the complex emergent behavior** induced on the system by the small level of noise considered. Interestingly enough, in the limit of **vanishing threshold**, q_i approaches unity, i.e., the **behavior becomes exponential**, which is precisely what was systematically observed in financial data*.

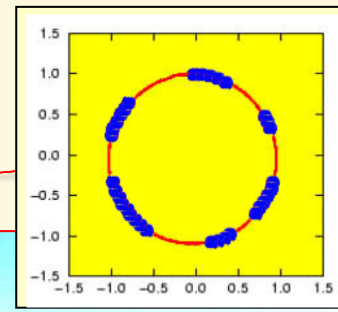
*J. Ludescher, C. Tsallis and A. Bunde, Europhys. Letters 95, 68002 (2011)



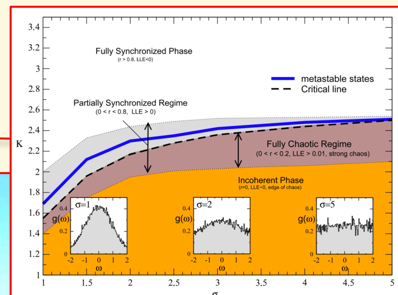
2002

work-time-line

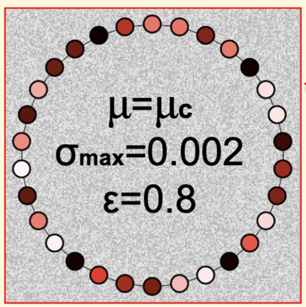
2004-2006



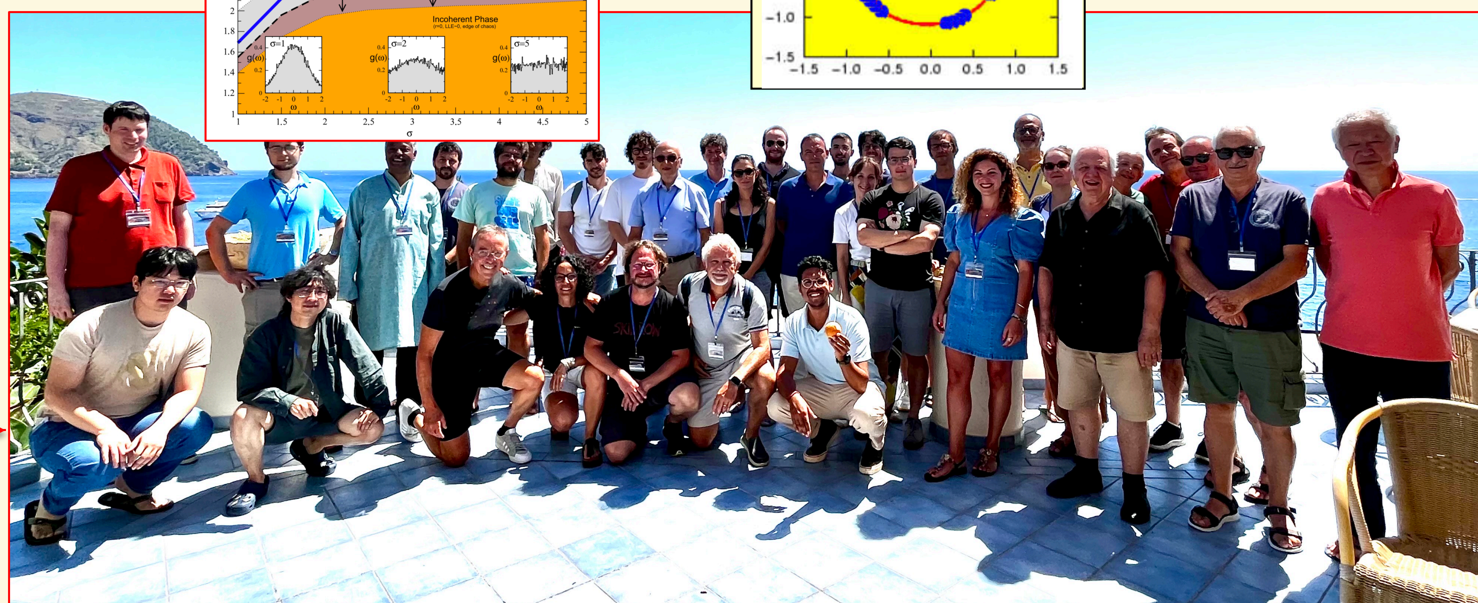
2009



2013

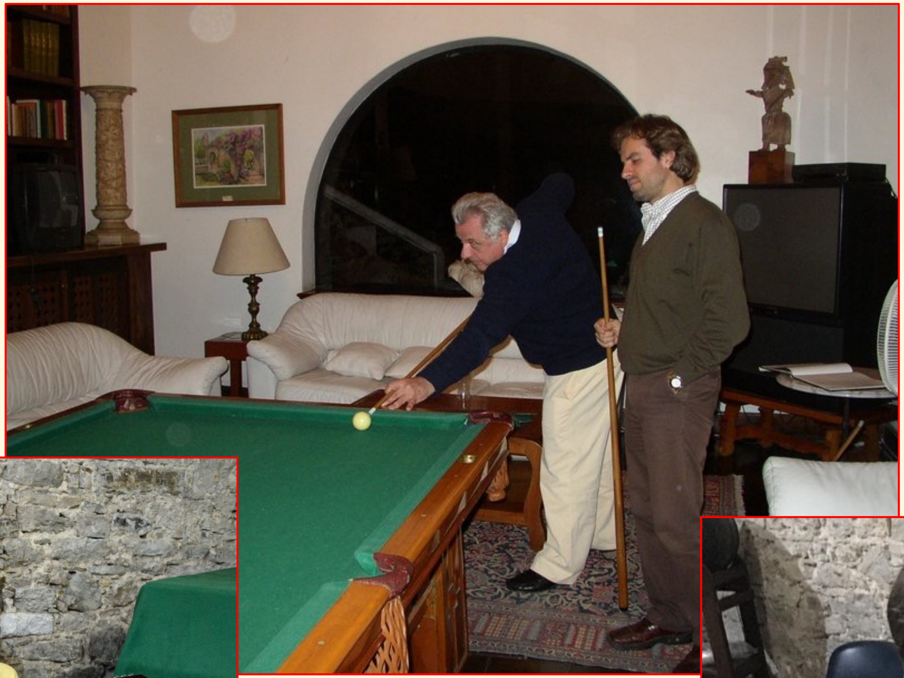


2023



Other steps of more pleasant work...;-)

Verifying the stability of chaotic trajectories with a Non-Sinai Billiard in Mexico



Studying the effects of acoustic emissions on the walls of the Ettore Majorana Center in Erice (Iyaly)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