Subspace of Hamiltonian's eigenfunctions described by the Generalized Heisenberg Algebra

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1. Introduction

The development of algebras aimed at quantum systems emerged in an approach to one-dimensional integrable quantum models using Bethe's algebraic ansatz. Then, the construction of different algebras was relevant to address a large class of physical phenomena. When the algebra is associated with the harmonic oscillator, it is called Heisenberg algebra and is presented in terms of the creation and annihilation operators. Its generalization (q-oscillators) was implemented through the group $su_q(2)$ in the Jordan-Schwinger method, which is a deformed version of the group $su(2)$. In this context, the generalized Heisenberg algebra (GHA) was forged in early 2000 [**1**,**2**]. It consists of a Heisenberg-type algebra that relies on an arbitrary function f. This function depends on the dimensionless Hamiltonian, and it is carefully chosen to correspond to the desired quantum system. Therefore, it is possible to uncover ladder operators and construct coherent states for several quantum systems. The main purpose of this work is to develop a generalized Heisenberg algebra for quantum billiards. Henceforth, we depict a subspace of Hamiltonian's eigenfunction via GHA. We find the position representation of the ladder operators, coherent states, and the respective time evolution.

2. GHA

The GHA is generated by the operators $\hat{H}, \hat{A}, \hat{A}^\dagger$ satisfying

$$\hat{H}\hat{A}^\dagger = \hat{A}^\dagger f(\hat{H}) \quad (1)$$

$$\hat{A}\hat{H} = f(\hat{H})\hat{A} \quad (2)$$

$$\left[\hat{A}, \hat{A}^\dagger\right] = f(\hat{H}) - \hat{H}, \quad (3)$$

where $\hat{A} = (\hat{A}^\dagger)^\dagger$ and the Hamiltonian is hermirtian. Along these lines, we set the general vector $|l,m\rangle$ (which is an eigenstate of the Hamiltonian) to depict the Fock space n-dimensional representation theory. Therefore

$$\hat{H}|l,m\rangle = \epsilon_{l,m}|l,m\rangle, \quad (4)$$

where m and l are quantum numbers, $\epsilon_{l,m}$ is the energy eigenvalue. It is worth mentioning that each state can be written as a tensor product of two Fock spaces $|l\rangle = |l\rangle \otimes |m\rangle$. We construct an algebra for each quantum number l. Therefore, for each algebra, we define the characteristic function as f_l, and $f_l^{(m)}(\epsilon_{l,1}) = \epsilon_{l,m}$, where $f_l^{(m)}$ is the m-th iterate of $\epsilon_{l,1}$. Hence, the operators $\hat{A}_l$ and $\hat{A}_l^\dagger$ are defined for each specific l, and its action on a general vector is

$$\hat{A}_l^\dagger|l,m\rangle = N_{l,m}|l,m+1\rangle \quad (5)$$

$$\hat{A}_l|l,m\rangle = N_{l,m-1}|l,m-1\rangle, \quad (6)$$

where $N_{l,m}^2 = \epsilon_{l,m+1} - \epsilon_{l,1}$.

These generalized Heisenberg algebras describe several classes of quantum systems. It was presented in [**?**] that the class is characterized by quantum systems with the following relation

$$\epsilon_{l,m+1} = f_l(\epsilon_{l,m}), \quad (7)$$

where both ϵ are consecutive eigenvalues, f_l is the characteristic function in eq. (3), and $\hat{A}_l$ and $\hat{A}_l^\dagger$ are the annihilation and creation operators.

3. Coherent states

The GHA formulation provides an explicit form to obtain Klauder-type coherent states [**3**]

$$\hat{A}_l|z,l\rangle = z|z,l\rangle, \quad (8)$$

where z is a complex number. Thus, one can expand $|z,l\rangle = \sum_{m=1}^{\infty} c_{l,m}|l,m\rangle$, and perform the action of the annihilation operator together with the eq. (6) and (8)

$$\hat{A}_l|z,l\rangle = \sum_{m=1}^{\infty} c_{l,m+1}N_{l,m}|l,m\rangle = z\sum_{m=1}^{\infty} c_{l,m}|l,m\rangle, \quad (9)$$

therefore, the coefficients $c_{l,m}$ is obtained via

$$c_{l,m} = c_{l,1}\frac{z^m}{N_{l,m-1}!}, \quad (10)$$

where $N_{l,m}! = N_{l,1}N_{l,2}\dots N_{l,m}$. Defining $N_l(z) = c_{l,1}$ and $N_{l,0}! = 1$ for consistency, one can write the coherent states as

$$|z,l\rangle = N_l(|z|)\sum_{m=1}^{\infty}\frac{z^m}{N_{l,m-1}!}|l,m\rangle. \quad (11)$$

The Klauder's coherent states are obtained by the minimal set of conditions, which are the normalizability $\langle z|z\rangle = 1$, continuity in the label

$$|z - z'| \longrightarrow 0, \qquad \||z\rangle - |z'\rangle\| \longrightarrow 0, \quad (12)$$

and completeness

$$|l\rangle\langle l| \otimes \hat{\mathbb{I}}_m = \int d^2z\omega(z)|z,l\rangle\langle z,l|, \quad (13)$$

where $\omega(z)$ is a weight function, and in the left hand side of eq.13 $\hat{\mathbb{I}}_m = \sum_m |m\rangle\langle m|$ is the identity operator. In this context, it is possible to construct coherent states belonging to a subset of all the eigenstates. From the normalizability condition, one can obtain

$$N_l(|z|) = \left[\sum_{m=1}^{\infty}\frac{|z|^{2m}}{N_{l,m-1}^2!}\right]^{-1/2}, \quad (14)$$

and from the completeness condition, it is possible to find the weight function $\omega(z)$, for given spectra. We can show the characteristic function of the algebra and construct the coherent states with one of the quantum numbers l constant. We will consider the eigenvalues written as a second-order polynomial $\epsilon_{l,m} = am^2 + bm + c$, where a, b, and c depends of l.

4. Quantum Billiards

Quantum billiards refer to the study of quantum mechanical systems where a particle is confined within a bounded region. We select three representative cases: the square billiard, the circular billiard, and the equilateral triangle. Both square and circular billiards are defined as separable billiards. The meaning behind this classification is that their eigenstates can be found via the separation of variables method. The equilateral triangle is an example of a nonseparable billiard. Also, a second-order polynomial gives the eigenvalues of the square and the equilateral billiard. On the other hand, the circular billiard eigenvalues do not obey this specific function. However, we can treat it as a second-order polynomial via a suitable approximation. Along these lines, we determine the algebra generators and their position representation, we construct coherent states, and verify their quantum time revival.

4.1 Square Billiard

The dynamics of a particle confined in a rectangular billiard are governed by the Schrodinger equation with the potential

$$V(\mathbf{r}) = \begin{cases} 0 & 0 < x < L_x \text{ and } 0 < y < L_y \\ \infty & \text{otherwise} \end{cases}, \quad (15)$$

therefore the particle can not be found outside of the rectangle of sides L_x and L_y. Following the separation of variables method, one can find the energy eigenvalues

$$\epsilon_{l,m} = \frac{\hbar^2\pi^2}{2\mu}\left(\frac{l^2}{L_x^2} + \frac{m^2}{L_y^2}\right), \quad (16)$$

where $l, m = 1, 2, 3...$ associated with the eigenfunctions

$$\psi_{l,m}(\mathbf{r}) = \frac{2}{\sqrt{L_xL_y}}\sin\left(\frac{l\pi x}{L_x}\right)\sin\left(\frac{m\pi y}{L_y}\right). \quad (17)$$

Setting $\hbar = 2\mu = L_x = L_y = 1$, we depict the square billiard. In this context, we associate the eigenfunction $\psi_{l,m}$ in the Hilbert space with each state $|l,m\rangle$ in Fock state space. Then, one can easily see that the characteristic function is

$$f_l(\epsilon_{l,m}) = \pi^2 + \epsilon_{l,m} + \sqrt{4\pi^2\epsilon_{l,m} - 4l^2\pi^4}. \quad (18)$$

Hence, the action of the algebra generators are

$$\hat{H}|l,m\rangle = \pi^2(l^2 + m^2)|l,m\rangle, \quad (19)$$

$$\hat{A}_l^\dagger|l,m\rangle = \pi\sqrt{m(m+2)}|l,m+1\rangle, \quad (20)$$

$$\hat{A}_l|l,m\rangle = \pi\sqrt{m^2 - 1}|l,m-1\rangle, \quad (21)$$

with ladder operators in the position representation

$$\hat{A}_l^\dagger\psi_{l,m}(\mathbf{r}) = \left[g_1(y)\frac{d}{dy} + g_2(y)\rho(\hat{N})\right]\tau(\hat{N})\psi_{l,m}(\mathbf{r}) = N_{l,m}\psi_{l,m+1}(\mathbf{r}) \quad (22)$$

where

$$g_1(y) = \sin(\pi y) \quad (23)$$

$$g_2(y) = \cos(\pi y) \quad (24)$$

$$\rho(\hat{N}) = \hat{N}\pi \quad (25)$$

$$\tau(\hat{N}) = \frac{\sqrt{\hat{N}(\hat{N}+2)}}{\hat{N}}, \quad (26)$$

and

$$\hat{A}_l = \tau(\hat{N})\left[-\frac{d}{dy}g_1(y) + \rho(\hat{N})g_2(y)\right], \quad (27)$$

and $\hat{N}$ is the number operator. So, it yields the commutation relations

$$\left[\hat{H}, \hat{A}_l^\dagger\right] = \hat{A}_l^\dagger\left(\pi^2\mathbb{I} + 2\pi\sqrt{\hat{H} - l^2\pi^2}\right) \quad (28)$$

$$\left[\hat{H}, \hat{A}_l\right] = -\left(\pi^2\mathbb{I} + 2\pi\sqrt{\hat{H} - l^2\pi^2}\right)\hat{A}_l \quad (29)$$

$$\left[\hat{A}_l, \hat{A}_l^\dagger\right] = \pi^2\mathbb{I} + 2\pi\sqrt{\hat{H} - l^2\pi^2}. \quad (30)$$

In position representation, we have the time evolution of a coherent state

$$\mathcal{Z}_l(\mathbf{r}, z, t) = \frac{1}{\sqrt{I_2(2|z|/\pi)}}\sum_{m=1}^{\infty}\left(\frac{z}{\pi}\right)^m\frac{e^{-it\pi^2(l^2+m^2)}}{\sqrt{(m-1)!(m+1)!}}\psi_{l,m}(\mathbf{r}), \quad (31)$$

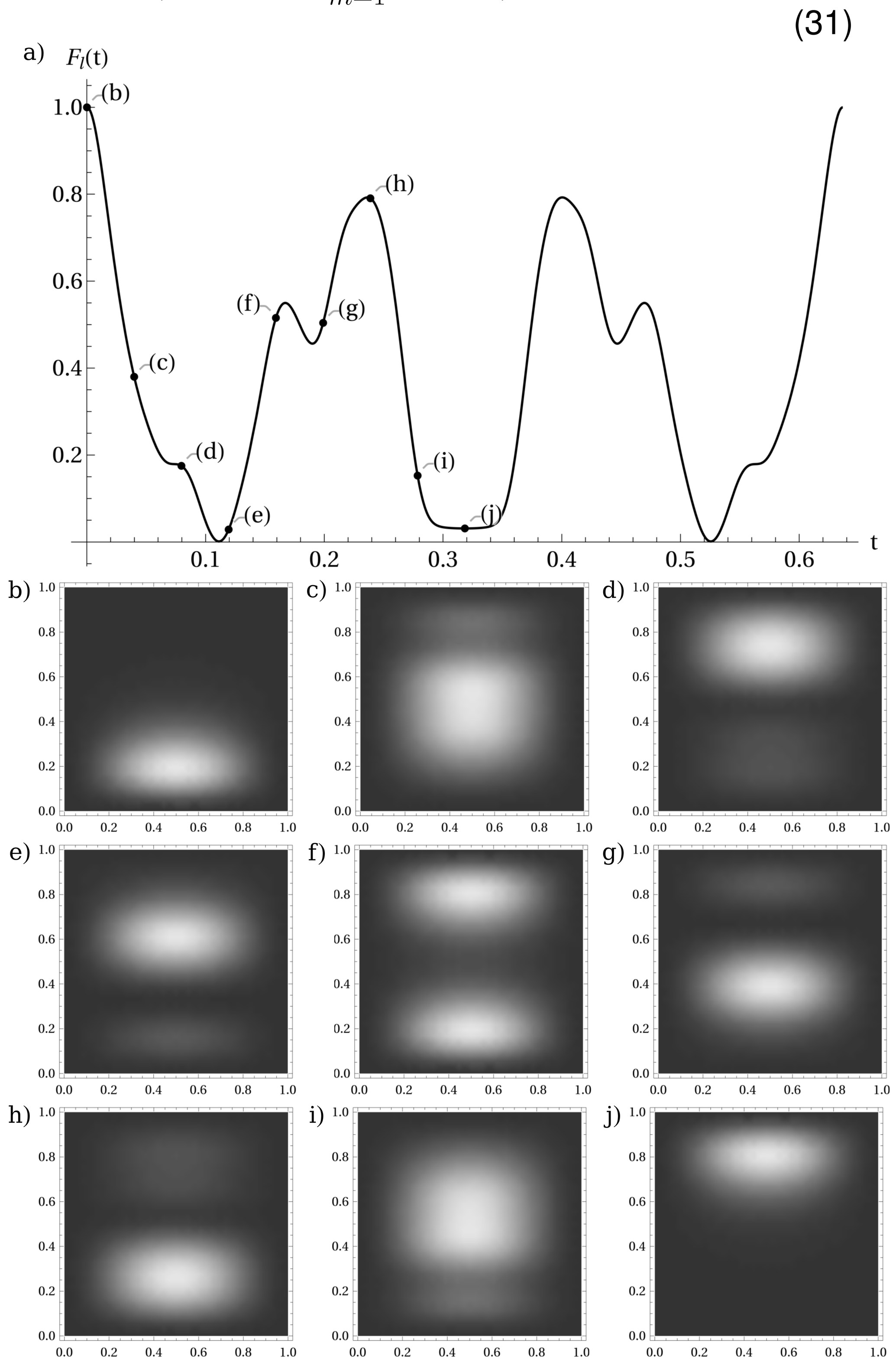

Figure 1: *Plot of the fidelity (a) between the initial coherent state $|z,1\rangle$ and the evolving one $|z,1,t\rangle$ with $l = 1$ over a quantum revival time T_{rev}. The dots indicate the time for each specific density plot (b-j) of the probability density of the time-evolved one-dimensional coherent state in position representation $|\mathcal{Z}(\mathbf{r}, z, t)|^2$ (eq 31), where $z = 5$.*

5. References

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