Dynamics in Fractal Spaces

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University of São Paulo
Brazil

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CBPF - Rio de Janeiro

Fractal Derivatives

Haussdorff Geometry:

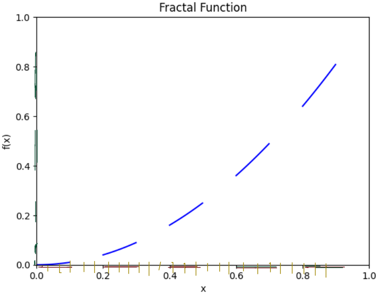
K. Falconer: Fractal Geometry: Mathematical Foundations and Applications

δ-measure - $\mathcal{H}^\alpha_\delta(\mathbb{F}) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^\alpha : \{U_i\} \text{is a cover of } \mathbb{F}\right\}$

Haussdorff measure - $\mathcal{H}^\alpha(\mathbb{F}) = \lim_{\delta\to 0} \mathcal{H}^\alpha_\delta(\mathbb{F})$

If $\mathbb{F}$ is a Borel set in $\mathbb{R}^\alpha \Rightarrow \mathcal{H}^\alpha(\mathbb{F}) = c_\alpha^{-1} vol^\alpha(\mathbb{F})$

Mass distribution $\gamma^\alpha_{\mathbb{F}}(a,b) = \mathcal{H}^\alpha([a,b])$ if $0 < \mathcal{H}^\alpha < \infty$



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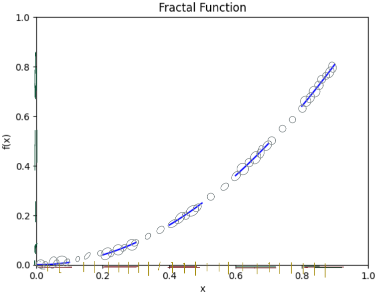

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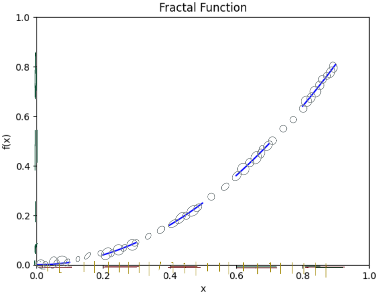

FractalDerivative

Parvate & Gangal, Fractals 17 (2009) 53

$$D^{\alpha}_{\mathbb{F},a} f(x_o) = \begin{cases} F\lim_{x\to x_o} \frac{f(x)-f(x_o)}{S^{\alpha}_{F,a_o}(x)-S^{\alpha}_{F,a_o}(x_o)} \\ 0 \quad \text{otherwise} \end{cases}$$

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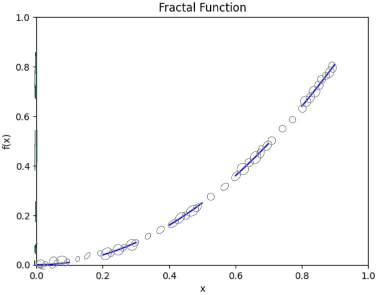

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$$D^\alpha_{\mathbb{F},\varphi} f(x_o) = \begin{cases} F\lim_{x\to x_o} \dfrac{S^\alpha_{F,\varphi}(f_x)-S^\alpha_{F,\varphi}(f_{x_o})}{x-x_o} \\ 0 \quad \text{otherwise} \end{cases}$$

AD, E. Megías & R. Pasechnik, Entropy 25 (2023) 1008

Continuous Approximations and Fractional Derivatives

Continuous Approximation:

$dS^{\alpha}_{F,\varphi}(x) = S^{\alpha}_{F,\varphi}(x+dx) - S^{\alpha}_{F,\varphi}(x)$

$dS^{\alpha}_{F,\varphi}(x)$ is the volume of the ball at $x+dx$.

$$dS^{\alpha}_{F,\varphi}(x) = \begin{cases} A(\alpha)x^{\alpha-1}dx \text{ if } x, x+dx \in \mathbb{F} \\ 0 \text{ otherwise} \end{cases} .$$

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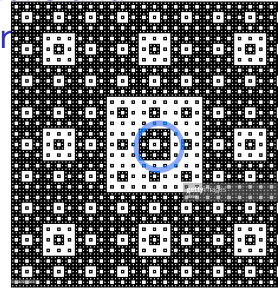

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Fractal Derivatives

In domain space

$D_{conf}\, f(x) = x^{1-\alpha}\frac{df}{dx}(x)$

$D_C\, f(x) = \int_{x_o}^{x} (x-t)^{1-\alpha}\frac{df}{dt}(t)dt$

In image space

$D_q f(x) = f^{1-q}\frac{df}{dx}(x)\,, \quad \alpha = 2-q$

$D_{Cl} f(x) = \int_{x_o}^{x} [f(x)-f(t)]^{1-q}\frac{df}{dt}dt$

AD, E. Megías & R. Pasechnik, Entropy 25 (2023) 1008 and arXiv:2305.04633

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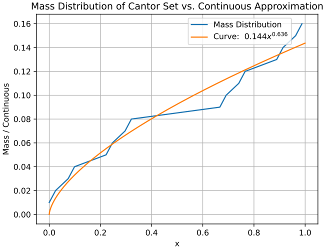

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Dynamics in the fractal medium

Fokker-Planck Equation

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial p_i}\left[A_i f + \frac{\partial (B_{ij} f)}{\partial p_j}\right] = 0$$

$$f(\mathrm{p}, t) = \exp\left[-\frac{(\mathrm{p}-\mathrm{p}_M(t))^2}{2\sigma(t)^2}\right]$$

Plastino-Plastino Equation:

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Thermofractal $\rightarrow$ Tsallis Statistics — AD PRD 2016

Yang-Mills fields $\rightarrow$ Thermofractals — AD, E. Megias, D.P. Menezes PRD 2020

$$\frac{1}{q-1} = \frac{11}{3} N_c + \frac{4}{3}\frac{N_f}{2}$$

$q = 8/7 = 1.14$ if $(N_c, N_f) = (3, 6)$

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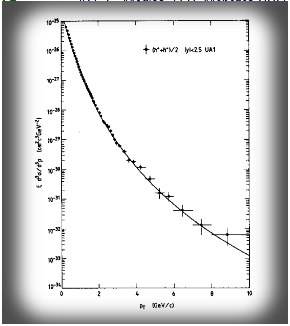
Experimental result: $q = 1.14 \pm 0.01$



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Walton, Rafelski PRL 2000

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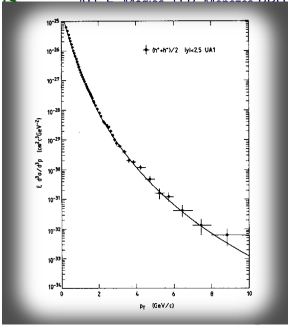

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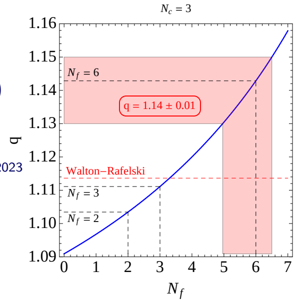
E. Megias, AD, R. Psachnik and C. Tsallis, PLB 2023



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Fractal Fokker-Planck Equation

$$[D^{\zeta}_{F',t}]h(\mathrm{p},t) = [D^{\zeta}_{F,p_{o,i}}]\left(A_i h(\mathrm{p},t) + [D^{\zeta}_{F,p_{0,j}}](B_{ij} h(\mathrm{p},t))\right)$$

$$f(\mathrm{p},t) = \frac{(\lambda\alpha)^d}{\left(\sqrt{2}\sigma_F(t)\right)^{d/\alpha}\Gamma\left[\frac{1}{2\alpha}\right]^d}\exp\left(-\frac{(\mathrm{p}^\alpha - \mathrm{p}_M^\alpha(t))^2}{2\sigma_F(t)^2}\right)$$

E. Megias, A. Golmankhaneh and A.D. in arxiv: 2309.13627

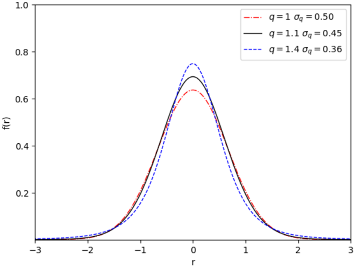
$\sigma_\infty \propto N^{1/2}$ $\sigma_{q;\infty} \propto N^{1/[2-d(q-1)]}$ $\sigma_{F;\infty} \propto N^{1/(2-\delta d_f)}$

$$N = \int d^d f(\mathrm{p}, t)$$

Dynamics in the fractal medium

$\frac{\partial f}{\partial t}$

$f(p,$



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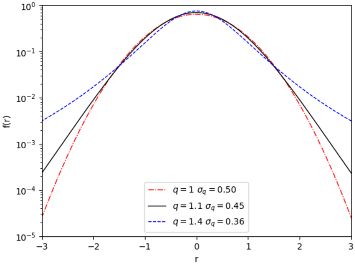

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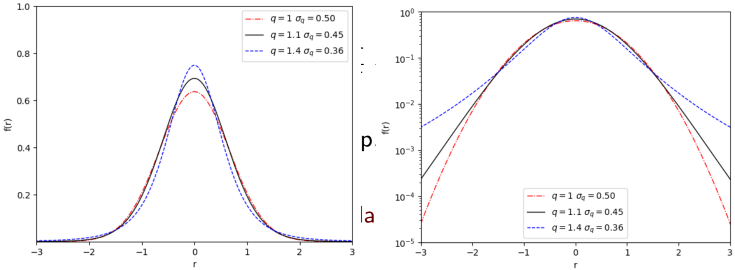

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The emerging Science of Cities:

F Xu et al. Research Square 2021
L. Bettencourt Science
FL Ribeiro, D Rybski, Phys. Rep. 2023

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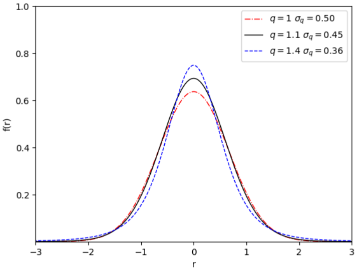

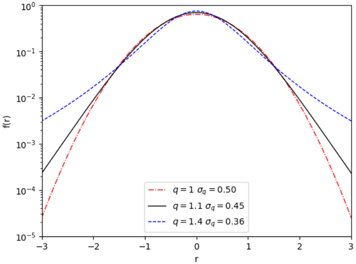

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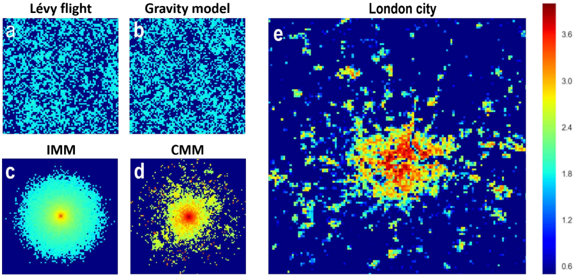

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$\sigma_{\infty} \propto l$

Conclusion

Many links between Tsallis Statistics and Fractal Geometry

Unveiled fractal aspects of QGP

Entropic parameter related to QFT parameters

Perspectives for application in other areas: City Dynamics.

See you in Tsallis' 90 !